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Stress–energy–momentum tensors in higher order variational calculus

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Abstract

Given a variational problem defined by a natural Lagrangian density $\mathcal{L}\omega$ on the k-jet extension $J^k(Y/X)$ of a natural bundle $p : Y \to X$ over an n-dimensional manifold X, oriented by a volume element ω, a stress–energy–momentum tensor $T(s)$ is constructed for each section $s \in \Gamma(X, Y)$ from the multimomentum map $\mu_\Theta : \Gamma(X, Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Omega^{n-1}(X))$ associated to any Poincaré–Cartan form Θ and to the natural lifting of vector fields $\mathfrak{X}(X)$ to the bundle $Y \to X$. The characterization made for $T(s)$ gives an intrinsic expression of this tensor as well as a generalization of the classical Belinfante–Rosenfeld formula. This tensor satisfies the typical properties of a stress–energy–momentum tensor: Diff(X)-covariance, Hilbert formula, conservation law, etc. Furthermore, it plays the expected role in the theory of minimal gravitational interactions.

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1. Introduction

In an inspired paper by Gotay and Marsden [14], a new method of constructing stress–energy–momentum tensors is presented, based on the Noether theory for first order variational calculus.

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Let Θ be the Poincaré–Cartan form associated to a Lagrangian density $\mathcal{L}\omega$ on the 1-jet extension $J^1(Y/X)$ of a bundle $p : Y \to X$ over an n-dimensional manifold X, oriented by a volume element ω. If the bundle $p : Y \to X$ and the Lagrangian density $\mathcal{L}\omega$ are natural, the multimomentum map $\mu : \Gamma(X, Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Omega^{n-1}(X))$ associated to the natural lifting $D \in \mathfrak{X}(X) \mapsto \tilde{D} \in \mathfrak{X}(Y)$ of vector fields $D \in \mathfrak{X}(X)$ by infinitesimal symmetries of the variational problem is given by the formula

$$\mu(s)(D) = (j^1_s)^*[i_{(j^1\tilde{D})}\Theta], \tag{1.1}$$

where $s \in \Gamma(X, Y)$.

In this situation a fundamental result of the above mentioned paper is the following:

If the natural lifting is of differential index 1, then for each section $s \in \Gamma(X, Y)$ there exists a unique tensor $\mathcal{T}(s) \in \Gamma(\bigwedge^1 T^*X \otimes \bigwedge^{n-1} T^*X)$ such that

$$\int_S i_D\mathcal{T}(s) = \int_S \mu(s)(D)$$

for each vector field of compact support $D \in \mathfrak{X}_C(X)$ and every hyper-surfaces $S \subset X$.

The tensor thus defined satisfies the typical properties of a stress–energy–momentum tensor, hence obtaining a variational characterization of this basic concept.

A natural question that now arises is the following: Is it possible to generalize this characterization to higher order variational calculus?

As is well known, the modern formulation of the Hamilton–Cartan theory of variational calculus, established in the early 1970s for first order problems [8,11,12,15], was generalized to higher order in the early 1980s [6,7,10,16]. In the diverse approaches proposed, the object canonically associated to a Lagrangian density $\mathcal{L}\omega$ on the k-jet extension $J^k(Y/X)$ is not a single Poincaré–Cartan form but rather a family $\{\Theta\}_{\mathcal{L}\omega}$ of them. In this way, under the same naturality conditions for the bundle $p : Y \to X$ and the Lagrangian density $\mathcal{L}\omega$, we have a family $\{\mu_\Theta\}_{\mathcal{L}\omega}$ of multimomentum maps defined by the family $\{\Theta\}_{\mathcal{L}\omega}$ of Poincaré–Cartan forms using the same formula (1.1) of the first order problems.

In the present work, the above result is generalized to higher order variational problems under the following differential formulation:

Theorem 1.1. *Given a natural Lagrangian density $\mathcal{L}\omega$ on the k-jet bundle $J^k(Y/X)$ of a natural bundle $p : Y \to X$ of differential index* 1, *one has:*

1. *For each section $s \in \Gamma(X, Y)$ there exists a unique tensor $\mathcal{T}(s) \in \Gamma(T^*X \otimes \bigwedge^{n-1} T^*X)$ such that*

$$i_D\mathcal{T}(s) = \mu_\Theta(s)(D) + \mathrm{d}\alpha \tag{1.2}$$

for every Poincaré–Cartan form Θ and every vector field $D \in \mathfrak{X}(X)$, where α is a $(n-2)$-form on X that depends on Θ, s and D.

2. *If $\mathcal{E} : s \in \Gamma(X, Y) \mapsto \mathcal{E}(s)$ is the Euler–Lagrange operator of the variational problem and P_s is the first order $\mathbb{R}$-linear differential operator between the vector bundles TX*

and $s^*V(Y)$ defined by $P_s : D \mapsto \tilde{D}^v_s$ ($\tilde{D}^v_s$ = p-vertical component of $\tilde{D}$ along s), then the tensor $\mathcal{T}(s)$ can be explicitly expressed by the formula

$$i_D\mathcal{T}(s) = -\mathcal{E}(s)(\sigma(P_s)(D)), \quad D \in \mathfrak{X}(X), \tag{1.3}$$

where $\sigma(P_s)$ is the symbol of the operator P_s and the contractions are the obvious ones.

3. *The assignation $s \in \Gamma(X, Y) \mapsto \mathcal{T}(s) \in \Gamma(T^*X \otimes \bigwedge^{n-1} T^*X)$ is $\mathrm{Diff}(X)$-covariant; that is, for every diffeomorphism $\varphi : X \to X$, it holds that*

$$\mathcal{T}(\tilde{\varphi}^*s) = \varphi^*\mathcal{T}(s),$$

where $\tilde{\varphi} : Y \to Y$ is the natural lifting of φ to the bundle $p : Y \to X$.

The characterization of $\mathcal{T}(s)$ given by formula (1.2) is in our opinion an important intrinsic generalization to the higher order of the classical Belinfante–Rosenfeld formula, according to which the stress–energy–momentum tensor $\mathcal{T}(s)$ obtained by adding the operator $D \mapsto d(\alpha)(\Theta, s, D))$ (the Belinfante–Rosenfeld "correction terms" in this framework) to the value at s of the multimomentum map $\mu_\Theta(s) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X})(X), \Omega^{n-1}(X))$. Moreover, formula (1.3), which expresses the tensor $\mathcal{T}(s)$ in terms of the Euler–Lagrange operator and the symbol of the operator defined by the natural lifting, should be interpreted as a generalization of the well-known Hilbert definition for this concept. Thus our formalism not only provides a general definition for the stress-energy–momentum tensor of a variational problem but also explains in a very simple and transparent way the relation existing between the two main historical approaches to this theory.

The plan of the paper is the following. In Section 2 we review the aspects of higher order variational calculus that will be used. In Section 3, we prove the theorem characterizing the stress–energy-momentum tensor, and its first properties and applications are established. In Section 4 we discuss the important special case of the metric parametrized Lagrangian densities, proving other basic properties (Hilbert formula, conservation law, etc.). In order to illustrate this theory, in Section 5 we address two remarkable examples: the "electromagnetic field" and a certain class of "non-perfect relativistic fluids". Particularly novel is the second example, which allows us to formulate a general variational theory of dissipative relativistic hydrodynamics. Finally, in Sectin 6 we analyze the role played by the new concept in the theory of minimal gravitational interactions.

All manifolds, maps, tensors, etc. will be considered to be C^∞. The notion of bundle will be understood in an ample sense, that is: a C^∞-locally trivial surjective submersion $p : Y \to X$. Throughout the paper we will use differential calculus with values in vector bundles, following Ref. [19] without explicitly mentioning it.

2. A review on higher order variational calculus

Here, we summarize the aspects of higher order variational calculus that we shall use. We shall follow the formulation developed in [10,25]. For other approaches to this topic, see [1,7,13,20,29] and references therein.

Let $p : Y \to X$ be a bundle over an n-dimensional manifold X, oriented by a volume element ω. Let $J^k(Y/X)$ (briefly J^k) be the k-jet bundle of local sections of p and $p_k : J^k \to Y$, $\overline{p}_k : J^k \to X$ the canonical projections. For each $h > k$, let $\pi_{hk} : J^h \to J^k$ be the projection $\pi_{hk}(j^h_x s) = j^k_x s$. If dim $Y = n + m$ and (x_j, y_i), $1 \leq j \leq n$, $1 \leq i \leq m$ is a fibered local coordinate system for p, we shall denote by (x_j, y^i_α), $|\alpha| \leq k$ the natural induced coordinate system for J^k; that is: $y^i_\alpha(j^k_x s) = (\partial^{|\alpha|}/\partial x_1^{\alpha_1}, \ldots, \partial x_n^{\alpha_n})(y_i \circ s)(x)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 2.1. Given a section $s \in \Gamma(X, Y)$, the kth-order vertical differential of s at a point $x \in X$ is the linear mapping

$$(d^v_k s)_x : T_{j^{k-1}_x s} J^{k-1} \to V_{j^{k-1}_x s}(J^{k-1})$$

given by the following formula:

$$(d^v_k s)_x(D) = D - (j^{k-1} s \circ \bar{p}_{k-1})_*(D),$$

where $V(J^k)$ is the vertical bundle of the projection $\bar{p}_k$.

This notion allows us to define a 1-form θ^k on J^k with values in the induced vector bundle $V(J^{k-1})J_k$, by the rule

$$\theta^k_{j^k_x s}(D) = (d^v_k s)_x(\pi_{k,k-1_*} D).$$

This is the so-called structure 1-form of J^k, which is given locally by the expression

$$\theta^k = \sum_i \sum_{|\alpha|<k} \theta^i_\alpha \otimes \frac{\partial}{\partial y^i_\alpha},$$

where

$$\theta^i_\alpha = \mathrm{d}y^i_\alpha - \sum_j y^i_{\alpha+(j)} \, \mathrm{d}x_j,$$

where (j) stands for the multiindex $(j)_k = \delta_{jk}$. This 1-form defines the basic structure of the jet bundles, from which the diverse notions of this theory are characterized. For example:

A section $\bar{s}$ of $\bar{p}_k : J^k \to X$ is the k-jet prolongation of a section s of p (i.e. $\bar{s} = j^k s$) if and only if $\bar{s}^*\theta^k = 0$.

A vector field $\bar{D}$ on J^k is an infinitesimal contact transformation of order k (i.c.t. of order k) if for any linear connection $\overline{\nabla}$ on $V(J^{k-1})$ there exists an endomorphism f of the vector bundle $V(J^{k-1})_{J^k}$ such that $L_{\bar{D}}\theta^k = f \circ \theta^k$, where the Lie derivative is taken with respect to the connection $\overline{\nabla}$. The previous condition does not depend on the connection $\overline{\nabla}$ and it holds that for any vector field D on Y (not necessarily p-projectable) there exists a unique i.c.t. of order k, $D_{(k)}$, projectable onto D. Moreover, the map $D \mapsto D_{(k)}$ is an injection of Lie algebras. The vector field $D_{(k)}$ is called the k-jet prolongation of the vector field D.

Locally, if $D = \sum_j u_j(\partial/\partial x_j) + \sum_i v_i(\partial/\partial y_i)$ with $u_j, v_i \in C^\infty(Y)$, then its k-jet extension is $D_{(k)} = \sum_j u_j(\partial/\partial x_j) + \sum_i \sum_{|\alpha|=0}^{k} v_\alpha^i(\partial/\partial y_\alpha^i)$, where the functions $v_\alpha^i \in C^\infty(J^{|\alpha|})$ are determined by the recurrence relations

$$v_0^i = v_i; \quad v_{\alpha+(l)}^i = \mathbb{D}_{k,l} v_\alpha^i - \sum_j y_{\alpha+(j)}^i \mathbb{D}_{k,l} u_j \ \ \forall |\alpha| < k,$$

where $\mathbb{D}_{k,l} = (\partial/\partial x_l) + \sum_i \sum_{|\alpha|<k} y_{\alpha+(l)}^i (\partial/\partial y_\alpha^i)$, $i \leq l \leq n$ is the total Lie derivative with respect to x_l, truncated at order k.

This characterization of the vector field prolongation admits the following more general version: if $k > 0$ and $\bar{D}$ is an i.c.t. of order k, for any $r > k$ there exists a unique i.c.t. of order r, $\bar{D}_{(r)}$, projectable onto $\bar{D}$. In particular, if D is a vector field on Y, it follows that $D_{(r)}$ is projectable onto $D_{(k)}$ for every $r > k$ (for more details on this issue, see [24]).

Henceforth we shall denote by $\mathfrak{X}^{(k)}$ the Lie algebra of the i.c.t. of order k and by $\mathfrak{X}_c^{(k)}$ the ideal of this algebra defined by the i.c.t. whose support has compact image in X by $\overline{p}_k$.

A variational problem of order k on the bundle $p : Y \to X$ is defined by a function $\mathcal{L} \in C^\infty(J^k)$. Then n-form $\mathcal{L}\omega$ (Lagrangian density) defines a functional $\mathcal{L} : S(X) \to \mathbb{R}$ by the rule

$$\mathbb{L}(s) = \int_{j^k s} \mathcal{L}\omega = \int_U (j^k s)^*(\mathcal{L}\omega),$$

where $S(X)$ is the space of sections $s : U \to Y$ (U an open set in X) for which this integral exists.

For any section $s \in \Gamma(X, Y)$ a linear form $\delta_s \mathbb{L} : \mathfrak{X}_c^{(k)} \to \mathbb{R}$ is defined by the rule

$$(\delta_s \mathbb{L})(\bar{D}) = \int_{j^k s} L_{\bar{D}}(\mathcal{L}\omega), \quad \bar{D} \in \mathfrak{X}_c^{(k)}. \tag{2.1}$$

Definition 2.2. A section s is critical for the Lagrangian density $\mathcal{L}\omega$ when $\delta_s \mathbb{L} = 0$.

A basic problem of the variational calculus is the characterization of critical sections as solutions of some differential system defined on an appropriate jet bundle. The Hamilton–Cartan formalism not only solves this problem, but also allows one to generalize the ideas and concepts from the ordinary analytical mechanics to variational calculus. The starting point is the notion of the "Poincaré–Cartan form", which, according to the formulation we are following, can be introduced thus

Theorem 2.3 [25]. *Given a Lagrangian density $\mathcal{L}\omega$ on J^r and a pair of linear connections $(\nabla, \overline{\nabla})$ on X and $V(Y)$, respectively, a $(V^* J^{k-1})_{J^{2k-1}}$-valued $(n-1)$-form Ω on J^{2k-1} (momentum form) can be univocally constructed, as well as a $(V^*Y)_{J^{2k-1}}$-valued n-form (Euler–Lagrange form $\mathbb{E}$) such that:*

$$\Theta = \theta^k \bar{\wedge} \Omega + \mathcal{L}\omega, \qquad \mathrm{d}\Theta = \theta^1 \bar{\wedge} \mathbb{E} + \theta^k \bar{\wedge} (\theta^{k-1} \bar{\wedge}_\eta), \tag{2.2}$$

where η is a $\mathrm{Hom}_{J^{2k-1}}(VJ^{2k-2}, V^*J^{k-1})$*-valued* $(n-1)$*-form on* J^{2k-1}*, and where the exterior products are taken with respect to the natural bilinear products.*

Interpreting this result as an existence theorem for n-forms Θ on J^{2k-1} satisfying the previous relations (2.2), we can give the following:

Definition 2.4. A Poincaré–Cartan form associated to the variational problem of Lagrangian density $\mathcal{L}\omega$ on the bundle J^k is an ordinary n-form Θ on J^{2k-1} such that there exist forms Ω, $\mathbb{E}$ and η satisfying conditions (2.2) of Theorem 2.3. The set of all these forms will be denoted by $\{\Theta\}_{\mathcal{L}\omega}$.

It should be mentioned that the different formulations of the Hamilton–Cartan theory of higher order variational calculus have as a starting point an existence theorem of the former kind. Conditions (2.2) characterizing the notion of a Poincaré–Cartan form are an intrinsic version of the classical "Lepage congruences" [21,22]. For a more detailed study of this notion, see [3,20,23].

Note that the family $\{\Theta\}_{\mathcal{L}\omega}$ is functorially associated to the Lagrangian density $\mathcal{L}\omega$ in the following sense: for every automorphism $\tau : Y \to Y$ of the bundle $p : Y \to X$, if $\tau_{(r)} : J^r$ is its r-jet prolongation it holds that $\tau^*_{(2k-1)}\Theta \in \{\Theta\}_{\tau^*_{(k)}\mathcal{L}\omega}$ for every $\Theta \in \{\Theta\}_{\mathcal{L}\omega}$.

The first application of the concept of the Poincaré–Cartan form is the characterization of critical sections and proof of the Noether theorem in the following way.

Given any Poincaré–Cartan form Θ, by deriving the first equality (2.2) with respect to an i.c.t. of order $2k-1$, $\bar{D}$ and bearing in mind the second of these equalities, we have the following.

Theorem 2.5 (*First variation formula*). *There exists a* $V^*(J^{2k-2})_{J^{2k-1}}$*-valued* $(n-1)$*-form* ξ *on* J^{2k-1} *such that*

$$L_{\bar{D}}(\mathcal{L}\omega) = \theta^1(\bar{D}) \circ \mathbb{E} + \mathrm{d}(i_{\bar{D}}\Theta) + \theta^{2k-1}\bar{\wedge}\xi, \quad \forall \bar{D} \in \mathfrak{X}^{(2k-1)}. \tag{2.3}$$

The linear functional $\delta_s\mathbb{L}$ defined by (2.1) will then be given by the formula

$$(\delta_s\mathcal{L})(D) = \int_{j^k s} L_D(\mathcal{L}\omega) = \int \theta^1(D)_{(2k-1)} \circ \mathbb{E}, \quad \forall D \in \mathfrak{X}_c^{(k)}.$$

Using Stokes' theorem and bearing in mind that the support of D has a compact image in X, it follows that:

Corollary 2.6 (*First characterization of critical sections*). *A section $s \in \Gamma(X, Y)$ is critical if and only if*

$$\mathbb{E}|_{j^{2k-1}s} = 0.$$

It is easy to see that, in a local coordinate system (x_j, y^i_α) on J^{2k-1}, we have

$$\mathbb{E}|_{j^{2k-1}s} = \sum_i \left(\sum_{r=0}^{k} \sum_{|\beta|=r} (-1)^r \left(\frac{\partial^{|\beta|}}{\partial x^\beta} \right) \left(\frac{\partial \mathcal{L}}{\partial y^i_\beta} \circ j^r s \right) \right) \mathrm{d}y_i \otimes \omega, \qquad (2.4)$$

which proves that the differential operator

$$\mathcal{E} : \Gamma(X, Y) \to \Gamma(X, s^* V^*(Y/X) \otimes \wedge^n T^* X)$$
$$s \mapsto \mathcal{E}(s) = \mathbb{E}|_{j^{2k-1}s}$$

does not depend on the particular Poincaré–Cartan form chosen and that it coincides locally with the classical Euler–Lagrange equations.

From the second equation in (2.2) it now follows, by Theorem 2.5, that:

Corollary 2.7 (*Second characterization of critical sections*). *A section $s \in \Gamma(X, Y)$ is critical if and only if for every vector field D on J^{2k-1} is holds that*

$$i_D \, \mathrm{d}\Theta|_{j^{2k-1}s} = 0.$$

Furthermore, this condition does not depend on the particular Poincaré–Cartan form chosen.

Regarding our second question, if we define an *infinitesimal symmetry* of a variational problem with Lagrangian density $\mathcal{L}\omega$ on J^k as an i.c.t. $\bar{D}$ of order k such that $L_{\bar{D}}\mathcal{L}\omega = 0$, the second equation in (2.2) and the second characterization of critical sections prove the following:

Theorem 2.8 (Noether). *If D is an infinitesimal symmetry of the variational problem given by the Lagrangian density $\mathcal{L}\omega$ on J^k, then for every Poincaré–Cartan form Θ and every critical section s, one has that*

$$\mathrm{d}(i_{\bar{D}_{(2k-1)}}\Theta)|_{j^{2k-1}s} = 0.$$

Returning to the construction of the forms Ω and $\mathbb{E}$ from a Lagrangian density $\mathcal{L}\omega$ and a pair of connections $(\nabla, \overline{\nabla})$ on X and $V(Y)$, respectively (Theorem 2.3), it should be noted that, for a given Lagrangian density, even though $\mathbb{E}$ depends on the two connections, Ω by contrast (and then $\Theta = \theta^r \bar{\wedge} \Omega + \mathcal{L}\omega$) only depends on the connection ∇.

Theorem 2.3 thus determines a subfamily of Poincaré–Cartan forms, parametrized by the set of linear connections on X, $\{\Theta_\nabla\}_{\mathcal{L}\omega} \subseteq \{\Theta\}_{\mathcal{L}\omega}$. Moreover, it can be demonstrated that for $k \geq 2$, the value taken by Θ_∇ at each point $j^{2k-1}_x s \in J^{2k-1}$ only depends on the jet J^{k-2}_x (sym ∇) of the symmetric connection associated with ∇. Therefore, if $\mathcal{K} \to X$ is the (affine) bundle of linear connections on the manifold X, we can give the following:

Definition 2.9. The universal Poincaré–Cartan form associated to a Lagrangian density $\mathcal{L}\omega$ on the jet bundle J^k is the ordinary n-form $\Theta(\mathcal{L}\omega)$ on the manifold $\mathcal{Z} = J^{k-2}(\mathcal{K}) \times_X J^{2k-1}(Y)$ given by the formula

$$(\Theta(\mathcal{L}\omega))_{(j_x^{k-2}\nabla, j_x^{2k-1}s)} = (\Theta_\nabla)_{j_x^{2k-1}s}.$$

This n-form, which depends only on the Lagrangian density, is universal with respect to the subfamily $\{\Theta_\nabla\}_{\mathcal{L}\omega} \subseteq \{\Theta\}_{\mathcal{L}\omega}$ in the sense that for each linear connection ∇ on X one has: $\Theta_\nabla = (j^{k-2}\nabla)^*\Theta(\mathcal{L}\omega)$.

The subfamily $\{\Theta_\nabla\}_{\mathcal{L}\omega}$ and the universal form $\Theta(\mathcal{L}\omega)$ are also functorially associated to the Lagrangian density $\mathcal{L}\omega$ in the following sense: For every automorphism $\tau : Y \to Y$ of the bundle $p : Y \to X$, if we denote by $\bar{\tau}$ the automorphism induced by the $\bar{p}$ projection of τ on the bundle $\mathcal{K} \to X$ of linear connections on X, then $\tau^*_{2k-1}(\Theta_{\bar{\tau}\nabla}(\mathcal{L}\omega)) = \Theta_\nabla(\tau^*_{(k)}\mathcal{L}\omega)$, and analogously, $(\bar{\tau}_{(k-2)}, \tau_{(2k-1)})^*\Theta(\mathcal{L}\omega) = \Theta(\tau^*_{(k)}\mathcal{L}\omega)$.

The family $\{\mathbb{E}_{\nabla,\overline{\nabla}}\}\mathcal{L}\omega$ of Euler–Lagrange forms depends on the pair $(\nabla, \overline{\nabla})$ as follows: if $\mathbb{E}$ and $\mathbb{E}'$ are the Euler–Lagrange forms corresponding to $(\nabla, \overline{\nabla})$ and $(\nabla', \overline{\nabla}')$, then there exists an $(n-1)$-form η on J^{2k-1} with values in $\mathrm{Hom}_{J^{2k-1}}(V(J^{2k-2}), V^*(Y))$-valued, horizontal over X, such that

$$\mathbb{E}' - \mathbb{E} = \theta^{2k-1}\bar{\wedge}\eta.$$

Accordingly, by the second equation in (2.2), for $k \geq 2$, if Θ and Θ' are the corresponding Poincaré–Cartan forms, there will exist a $\mathrm{Hom}_{J^{2k-1}}(VJ^{2k-2}, V^*J^{k-1})$-valued $(n-1)$-form $\bar{\eta}$ on J^{2k-1} such that

$$\mathrm{d}\Theta' - \mathrm{d}\Theta = \theta^k\bar{\wedge}(\theta^{2k-1}\bar{\wedge}\bar{\eta}).$$

Let us finally see how the corresponding Poincaré–Cartan forms differ from each other

$$\Theta' - \Theta = \theta^{k-1}\bar{\wedge}\psi,$$

where ψ is a $V^*(J^{k-2})_{J^{2k-1}}$-valued $(n-1)$-form on J^{2k-1}, horizontal over X.

If $\psi = \sum_{|\alpha|<k-1}(-1)^j\psi^i_{\alpha j}\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n \otimes \mathrm{d}y^i_\alpha$ is the local expression of the $(n-1)$-form ψ with respect to the local coordinate system (x_j, y^i_α), it is proved in [25] that the coefficients $\psi^i_{\alpha j}$ verify the equations

$$\sum_{\beta+(j)=\alpha}\psi^i_{\beta j} = 0, \quad |\alpha| = k-1, \qquad \sum_j \mathbb{D}_j\psi^i_{0j} = 0,$$
$$\sum_j \mathbb{D}_j\psi^i_{\alpha j} + \sum_{\beta+(j)=\alpha}\psi^i_{\beta j} = 0, \quad 0 < |\alpha| < k-1, \tag{2.5}$$

where $\mathbb{D}_j = \partial/\partial x_j + \sum_{i,\alpha} y^i_{\alpha+(j)}(\partial/\partial y^i_\alpha)$ is the total derivative with respect to x_j.

For $k = 2$, Eqs. (2.5) have as their only solution $\psi^i_{\alpha j} = 0$, which proves that in this case the family $\{\Theta_\nabla\}_{\mathcal{L}\omega}$ degenerates to a unique Poincaré–Cartan form. This form can be axiomatically characterized (see [26]). Its local expression in a set of unimodular coordinates with respect to ω is

$$\Theta = \sum_{i,j} \sum_{|\alpha|=0}^{1} f^i_{\alpha j} \theta^i_\alpha \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n + \mathcal{L}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \tag{2.6}$$

where

$$f^i_{(k)j} = (-1)^{j-1}\epsilon_{jk} \frac{\partial \mathcal{L}}{\partial y^i_{(jk)}},$$

$$f^i_{0k} = (-1)^{k-1} \frac{\partial \mathcal{L}}{\partial y^i_{(k)}} + \sum_j (-1)^k \epsilon_{jk} \mathbb{D}_j \left(\frac{\partial \mathcal{L}}{\partial y^i_{(jk)}} \right),$$

where $\epsilon_{jk} = 1$ if $j = k$, and $\epsilon_{jk} = \frac{1}{2}$ if $\neq k$, $(jk) = (j) + (k)$.

In particular, if $\mathcal{L} \in \mathcal{C}^\infty(J^1)$, the previous formula provides the usual Poincaré–Cartan form of first order variational calculus

$$\begin{aligned} \Theta = \theta^1 \bar{\wedge} \Omega + \mathcal{L}\omega = & \sum_i \left(\mathrm{d}y_i - \sum_k y^i_k \,\mathrm{d}x_k \right) \\ & \wedge \sum_j (-1)^{j-1} \frac{\partial \mathcal{L}}{\partial y^i_j} \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n \\ & + \mathcal{L}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n. \end{aligned} \tag{2.7}$$

For arbitrary k and $n = 1$, $\{\Theta\}_{\mathcal{L}\omega}$ also degenerates to a unique Poincaré–Cartan form, which coincides with that introduced in higher order analytical mechanics (see [30]).

3. The main theorem

From now on we shall assume that the bundle $p : Y \to X$ and the Lagrangian density $\mathcal{L}\omega$ on $J^k(Y/X)$ are natural [4,27,28]. From these concepts, here we shall use only the existence of a lifting $\varphi \mapsto \tilde{\varphi}$ to the bundle Y of the pseudo-group Diff(X) of the local diffeomorphisms of X by symmetries of the variational problem – that is: $\tilde{\varphi}^*_{(k)}\mathcal{L}\omega = \mathcal{L}\omega$ for every $\varphi \in$ Diff(X) – and its infinitesimal version: the existence of a lifting $D \mapsto \tilde{D}$ to the bundle Y of the real pseudo-Lie algebra $\mathfrak{X}(X)$ of the local vector fields on X by infinitesimal symmetries of the variational problem ($L_{\tilde{D}_{(k)}} = \mathcal{L}\omega = 0$ for every $D \in \mathfrak{X}(X)$). Moreover, we shall also assume that the lifting $D \mapsto \tilde{D}$ has differential index 1, that is, locally we may write $u_j(\partial/\partial x_j) \mapsto u_j(\partial/\partial x_j) + v_i(\partial/\partial y_i)$, where $v_i = C^{hi}_k(\partial u_k/\partial x_h) + C^i_j u_j$, and $C^{hi}_k, C^i_j \in \mathcal{C}^\infty(Y)$. This condition (obviously not depending on the coordinate system chosen) holds in particular for the tensorial and spinorial bundles over X.

For each sections $s \in \Gamma(X, Y)$ the map $P_s : D \mapsto \tilde{D}^V_s$ ($\tilde{D}^V_s = \pi$-vertical component of $\tilde{D}$ along s) is a first order linear differential operator between the vector bundles $T(X)$ and $s^*V(Y)$. Its local expression is

$$u_j\frac{\partial}{\partial x_j}\mapsto\left(C_k^{hi}(x_l,y_m(x))\frac{\partial u_k}{\partial x_h}+\left(C_j^i(x_l,y_m(x))-\frac{\partial y_i(x)}{\partial x_j}\right)u_j\right)s^*\frac{\partial}{\partial y_i}. \tag{3.1}$$

where $y_m = y_m(x)$ are the equations of the section s in the local coordinate system under consideration. We also include in this class of operators those whose differential index is 0.

The symbol $\sigma(P_s)$ of the differential operator P_s (always considered as a first order operator) defines a section of the vector bundle $T^*(X)\otimes T^*(X)\otimes s^*V(Y)$, whose local expression is

$$\sigma(P_s)=C_k^{hi}(x_l,y_m(x))\,\mathrm{d}x_k\otimes\frac{\partial}{\partial x_h}\otimes s^*\frac{\partial}{\partial y_i}.$$

Therefore, if P_s is of differential index 0 (so $C_k^{hi}=0$), its symbol $\sigma(P_s)$ will be 0.

Given a section $\mathcal{E}\in\Gamma(X,s^*V^*(Y)\otimes\bigwedge^n T^*X)$ and a vector field $D\in\mathfrak{X}(X)$, let us evaluate the n-form $\mathcal{E}(P_s(D))$ obtained by duality pairing between the bundles $s^*V(Y)$ and $s^*V^*(Y)$. If $\mathcal{E}=(\mathcal{E}_i s^*\mathrm{d}y_i)\otimes\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$ is the local expression of $\mathcal{E}$ and we denote by ω_h the inner product $i_{\partial/\partial x_h}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$, by (3.1) we have

$$\begin{aligned}\mathcal{E}(P_s(D))&=\mathcal{E}_i\left(C_k^{hi}\frac{\partial u_k}{\partial x_h}+\left(C_j^i-\frac{\partial y_i}{\partial x_j}\right)u_j\right)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\&=\mathcal{E}_iC_k^{hi}\,\mathrm{d}u_k\wedge\omega_h+\mathcal{E}_i\left(C_j^i-\frac{\partial y_i}{\partial x_j}\right)u_j\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\&=-\frac{\partial\mathcal{E}_iC_k^{hi}}{\partial x_h}u_k\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n+\mathrm{d}(\mathcal{E}_iC_k^{hi}u_k\omega_h)\\&\quad+\mathcal{E}_i\left(C_j^i-\frac{\partial y_i}{\partial x_j}\right)u_j\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\&=\left(-\frac{\partial}{\partial x_h}(C_j^{hi}\mathcal{E}_i)+\left(C_j^i-\frac{\partial y_i}{\partial x_j}\right)\mathcal{E}_i\right)u_j\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\&\quad+\mathrm{d}(\mathcal{E}(\sigma(P_s)(D))).\end{aligned}\tag{3.2}$$

It can therefore be defined a first order linear differential operator between the vector bundles $s^*V^*(Y)\otimes\bigwedge^n T^*X$ and $T^*X\otimes\bigwedge^n T^*X$ by the formula

$$P_s^-(\mathcal{E})(D)=\mathcal{E}(P_s(D))-\mathrm{d}[\mathcal{E}(\sigma(P_s)(D))] \tag{3.3}$$

Its local expression is given by the penultimate term in the calculation (3.2) it can immediately be seen that this operator coincides with the adjoint operator of P_s over the vector fields of compact support, that is

$$\int_X P_s^+(\mathcal{E})(D)=\int_X\mathcal{E}(P_s(D)) \tag{3.4}$$

for every $D\in\mathfrak{X}_c(X)$ and $\mathcal{E}\in\Gamma(X,s^*V^*(Y)\otimes\bigwedge^n T^*X)$.

It should also be remarked that the term in the differential in (3.3) defines, for each section $\mathcal{E}\in\Gamma(X,s^*V^*(Y)\otimes\bigwedge^n T^*X)$, a tensor $D\mapsto\mathcal{E}(\sigma(P_s)(D))$ on X, which will be essential in this work, and this is due to the fact that the lifting $D\mapsto\tilde{D}$ has differential index 1.

Finally, if $\Theta \in \{\Theta\}_{\mathcal{L}\omega}$ is a Poincaré–Cartan form of the variational problem, the multimomentum map

$$\mu_\Theta : \Gamma(X, Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Omega^{-1}(X))$$

associated to Θ and the natural lifting $D \mapsto \tilde{D}$ is given by the formula

$$\mu_\Theta(s)(D) = (j^{2k-1}s)^*(i_{\tilde{D}_{(2k-1)}}\Theta), \quad s \in \Gamma(X, Y), \quad D \in \mathfrak{X}(X). \tag{3.5}$$

With the concepts and notations so introduced above we are now in a position to establish the main result of this section.

Theorem 3.1. *Given a natural Lagrangian density $\mathcal{L}\omega$ on the k-jet bundle $J^k(Y/X)$ of a natural bundle $p : Y \to X$ of differential index* 1, *one has:*

1. *For each section $s \in \Gamma(X, Y)$ there exists a unique tensor $\mathcal{T}(s) \in \Gamma(T^*X \otimes \bigwedge^{n-1} T^*X)$ such that, for every Poincaré–Cartan form Θ and vector field $D \in \mathfrak{X}(X)$, one has*

$$i_D\mathcal{T}(s) = \mu_\Theta(s)(D) + \mathrm{d}\alpha, \tag{3.6}$$

where α is a $(n-2)$-form on X depending on Θ, s and D.

2. *The tensor $\mathcal{T}(s)$ is explicitly given by*

$$i_D\mathcal{T}(s) = -\mathcal{E}(s)(\sigma(P_s)(D)), \tag{3.7}$$

where $\mathcal{E}$ is the Euler–Lagrange operator of the variational problem, $\sigma(P_s)$ is the symbol of the operator P_s associated to the natural lifting $D \mapsto \tilde{D}$, and the contractions are the obvious ones.

3. *The assignation $s \in \Gamma(X, Y) \mapsto \mathcal{T}(s) \in \Gamma(T^*X \otimes \bigwedge^{n-1} T^*X)$ is* Diff(X)*-covariant, that is, for every diffeomorphism $\varphi : X \to X$ one has*

$$\mathcal{T}(\tilde{\varphi}^*s) = \varphi^*(\mathcal{T}(s)),$$

where $\tilde{\varphi} : Y \to Y$ is the natural lifting of the diffeomorphism φ to the bundle $p : Y \to X$.

Proof. *Uniqueness.* Let $s \in \Gamma(X, Y)$ If $\mathcal{T}(s)$ and $\mathcal{I}'(s)$ satisfy (3.6) every Poincaré–Cartan form Θ and vector field $D \in \mathfrak{X}(X)$, then, in particular, fixing Θ, the tensor $T = \mathcal{T}(s) - \mathcal{T}'(s) \in \Gamma(T^*X \otimes \bigwedge^{n-1} T^*X)$ satisfies that the $(n-1)$-form i_DT is closed for every D. Thus, for any function f and vector field D, we have $0 = \mathrm{d}(i_{fD}T) = \mathrm{d}f \wedge i_DT$.

Let $T_1^1(D)$ be the unique vector field such that $i_{T_1^1(D)}\omega = i_DT$ (ω being the volume element of X). Applying $i_{T_1^1(D)}$ to the equality $\mathrm{d}f \wedge \omega = 0$, we then have $T_1^1(D)(f) = 0$, and hence due to the arbitrariness of f, we have that $T_1^1(D) = 0$, and hence $i_DT = 0$. Accordingly, bearing in mind the arbitrariness of D, we have that $T = 0$, which implies $\mathcal{T}(s) = \mathcal{T}'(s)$.

Existence. We shall show that the tensor $\mathcal{T}(s)$ defined by formula (3.7) satisfies (3.6) for every Poincaré–Cartan form Θ and vector field $D \in \mathfrak{X}(X)$, thus proving the existence.

Since $\tilde{D}$ is an infinitesimal symmetry of the variational problem, by making use of the variation formula (2.3) and bearing in mind that $\theta^1(\tilde{D}_{(2k-1)}) \circ \mathbb{E}|_{j^{2k-1}s} = \mathcal{E}(s)(P_s(D))$ and formulas (3.3) and (3.4), we have

$$\begin{aligned} 0 = L_{\tilde{D}_{(k)}}\mathcal{L}\omega|_{j^k s} &= \mathcal{E}(s)(P_s(D)) + \mathrm{d}(i_{\tilde{D}_{(2k-1)}}\Theta)|_{j^{2k-1}s} \\ &= P_s^+(\mathcal{E}(s))(D) + \mathrm{d}(\mathcal{E}(s)(\sigma(P_s)(D)) + \mu_\Theta(s)(D)). \end{aligned} \tag{3.8}$$

Applying this equality to the vector fields D of compact support and integrating over the manifold X, it follows that

$$\int_X P_s^+(\mathcal{E}(s))(D) = 0, \quad \forall D \in \mathfrak{X}_c(X).$$

Therefore, for every section $s \in \Gamma(X, Y)$, we have

$$P_s^+(\mathcal{E}(s)) = 0. \tag{3.9}$$

Thus, returning to (3.8) the $(n-1)$-form $\omega_{n-1}^D = \mathcal{E}(s)(\sigma(P_s)(D)) + \mu_\Theta(s)(D)$ is closed. Moreover, let us see that it is exact, which concludes the proof of the first two assertions in Theorem 3.1.

Making use of the de Rham theorem, it would suffice to see that the integral of ω_{n-1}^D over the differential $(n-1)$-simplices of X vanishes. We shall prove that for any compact $(n-1)$-submanifold $S \subset X$, one has

$$\int_S \omega_{n-1}^D = 0.$$

Indeed, let $\{f_i\}$ be a partition of the unity subordinated to the cover $\{U_i\}$, $i = 1, \ldots, h+1$, where U_i are charts in which $S \cap U_i$ is a hyperplane and $U_{h+1} = X - S$. We then have $D = \sum f_i D$, and hence $\omega_{n-1}^D = \sum \omega_{n-1}^{f_i D}$, where $\omega_{n-1}^{f_i D}$ are closed $(n-1)$-forms of compact support contained in U_i. For each $i = 1, \ldots, h$ let V_i be a regular domain in U_i such that $\mathrm{Supp}(\omega_{n-1}^{f_i D}) \cap \partial V_i = \mathrm{Supp}(\omega_{n-1}^{f_i D}) \cap S$. In these conditions we have

$$\int_S \omega_{n-1}^{f_i D} = \int_{S \cap U_i} \omega_{n-1}^{f_i D} = \int_{\partial V_i} \omega_{n-1}^{f_i D} = \int_{V_i} \mathrm{d}\omega_{n-1}^{f_i D} = 0$$

and so

$$\int_S \omega_{n-1}^D = \sum_{i=1}^h \int_S \omega_{n-1}^{f_i D} = 0.$$

Diff(X)-*covariance*. By the naturality of $\mathcal{L}\omega$, we have $\tilde{\varphi}_{(k)}^*\mathcal{L}\omega = \mathcal{L}\omega$ and hence, bearing in mind the functoriality of the assignation $\mathcal{L}\omega \mapsto \{\Theta\}_{\mathcal{L}\omega}$, if Θ is a Poincaré–Cartan form of the variational problem, then $\tilde{\varphi}_{(2k-1)}^*\Theta$ is a Poincaré–Cartan form as well. Let $s \in \Gamma(X, Y)$; applying φ to (3.5), we have

$$i_{\varphi * D}[\varphi^*(\mathcal{T}(s))] = [j^{2k-1}(\tilde{\varphi}(s))]^*[i_{\widetilde{\varphi * D}(2k-1)}\tilde{\varphi}^*_{(2k-1)}\Theta] + \mathrm{d}\varphi^*\alpha$$
$$= [\mu_{\tilde{\varphi}^*_{(2k-1)}\Theta}(\tilde{\varphi}(s))](\varphi^* D) + \mathrm{d}\varphi^*\alpha$$

and then it follows that $\varphi^*\mathcal{T}(s) = \mathcal{I}(\tilde{\varphi}(s))$ by the uniqueness of $\mathcal{I}(\tilde{\varphi}(s))$. □

Remark 3.2. *If* $\dim X = n = 1$ *and* X *is connected, the former demonstration proves that in this case* $i_D\mathcal{T}(s) = \mu_\Theta(s)(D)$ *for every* $D \in \mathfrak{X}(X)$*; that is,* $\mu_\Theta(s)$ *is a* 1*-covariant* 1*-contravariant tensor on* X*, independent of the chosen Poincaré–Cartan form* Θ*, whose explicit expression is given by formula* (3.7).

For $\dim X = n > 1$ *the fundamental equation* (3.6) *characterizing the stress–energy–momentum tensor* $\mathcal{T}(s)$ *can be interpreted as a "Belinfante–Rosenfeld formula", from which this tensor is obtained by adding to the value* $\mu_\Theta(s) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Omega^{n-1}(X))$ *of the multimomentum map (which is not a tensor!) a "corrective term" given by the last terms in the formula mentioned.*

Another two very useful versions of the new concept are the following:

Using the volume element ω on X, $\mathcal{T}(s)$ can be expressed as a 1-covariant tensor $T_1^1(s)$ by the formula

$$i_{T_1^1(s)(D)}\omega = i_D\mathcal{T}(s), \qquad D \in \mathfrak{X}(X).$$

In a more explicit way, if we write $\mathcal{E}(s) = \mathcal{E}(s) \otimes \omega$ and note the T^*X-dependence on the symbol $\sigma(P_s)$, then by formula (3.7), we have

$$T_1^1(s)(D, \eta) = \mathcal{E}(s)(\sigma(P_s)(D, \eta)), \qquad D \in \mathfrak{X}(X), \quad \eta \in \Omega^1(X), \tag{3.10}$$

where the contraction on the right-hand side is that induced by duality between the vector bundles $s^*V(Y)$ and $s^*V^*(Y)$.

Furthermore, the assignation $s \mapsto \mathcal{T}(s)$ induces a tensor T on $J^{2k}(Y/X)$ by the rule

$$(i_{\bar{D}}T_{j_x^{2k}s})(\bar{D}^1, \ldots, \bar{D}^{n-1}) = (i_{(\bar{p}_{2k})_*\bar{D}}\mathcal{T}(s))((\bar{p}_{2k})_*\bar{D}^1, \ldots, (\bar{p}_{2k})_*\bar{D}^{n-1}),$$

where $\bar{D}, \bar{D}^1, \ldots, \bar{D}^{n-1} \in T_{j_x^{2k}s}(J^{2k})$ and s is a representative of the point $j_x^{2k}s \in J^{2k}(Y/X)$.

By construction, T is a tensor on J^{2k}, $\bar{p}_{2k}$-horizontal, skewsymmetric in the variables $\bar{D}^1, \ldots, \bar{D}^{n-1}$, invariant by the natural lifting $\varphi \in \mathrm{Diff}(X) \mapsto \tilde{\varphi}_{(2k)} \in \mathrm{Aut}(J^{2k})$, and fulfilling the universal property: $\mathcal{T}(s) = (j^{2k}s)^*T$ for each $s \in \Gamma(X, Y)$.

Thus, we associate in a canonical way a tensor $T_{\mathcal{L}\omega}$ on J^{2k} to the Lagrangian density $\mathcal{L}\omega$ on J^k with the previously mentioned properties, that can be called the "universal stress–energy–momentum tensor" of the variational problem.

By (2.4), its local expression is

$$T_{\mathcal{L}\omega} = -\left(\sum_{r=0}^{k}\sum_{|\beta|=r}(-1)^r\mathbb{D}^\beta\frac{\partial\mathcal{L}}{\partial y^i_\beta}\right)C_k^{hi}(\mathrm{d}x_k \otimes \omega_h),$$

where $\mathbb{D}^\beta = \mathbb{D}_1^{\beta_1} \circ \cdots \circ \mathbb{D}_n^{\beta_n}$, $\mathbb{D}_i$ being the total derivative with respect to x_i (observe that $[\mathbb{D}_i, \mathbb{D}_j] = 0$, which justifies the notation), and where we agree that $\mathbb{D}^\beta = \mathrm{Id}$ for $|\beta| = 0$.

A first application of the present formalism is the actual Noether theorem for these kinds of variational problem.

From (3.7) it follows that, if the section $s \in \Gamma(X, Y)$ is critical, then $\mathcal{T}(s) = 0$, thus obtaining (by (3.6))

$$\mu_{\Theta}(s)(D) = -\,\mathrm{d}\alpha, \quad D \in \mathfrak{X}(X). \tag{3.11}$$

That is, the restriction of the Noether invariants $i_{\tilde{D}_{2k-1}}\Theta$ to the critical sections are not only closed forms (Noether's theorem) but, moreover, exact.

Additionally, if the lifting $D \mapsto \tilde{D}$ fulfills the additional condition of being *vertically transitive* in the sense that for every point $j^1_x s \in J^1(Y/X)$ one has $\{\theta^1_{j^1_x s}(\tilde{D})/D \in \mathfrak{X}(X)\} = V_{s(x)}(Y)$, then the reciprocal holds, that is, the condition $\mathcal{T}(s) = 0$ (equivalently, Eq. (3.11)) implies that the section s is critical.

Accordingly, the critical sections $s \in \Gamma(X, Y)$ are the solutions of the equation $\mathcal{T}(s) = 0$. Or – modulus exact $(n-1)$-forms – the zero-level of the multimomentum map μ_{Θ}: $\Gamma(X, Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Omega^{n-1}(X))$. In particular, the latter observation is of relevant interest for the study of the presymplectic manifold of solutions of this kind of variational problem.

We will finish this section with two examples that illustrate the general result just obtained in a very clear way.

Example 1: First order problems. Taking the usual Poincaré–Cartan form (2.7) for these problems, a straightforward local calculation allows us to directly prove the equality

$$i_D\mathcal{T}(s) = \mu_{\Theta}(s)(D) + \mathrm{d}[\Omega(\sigma(P_s)(D))],$$

where Ω is the momentum $(n-1)$-form of the variational problem.

This is an attractive intrinsic expression of the Belinfante–Rosenfeld formula, in which the corrective term is calculated in a very simple way from the momentum form and the symbol of the natural lifting.

This gives, for our case, an alternative proof of the main theorem (Theorem 3.1), which is more in keeping with the proof given in [14].

Example 2: General relativity. As is well known, this theory is defined by a second order variational problem on the bundle $\pi : \mathcal{M} \to X_4$ of the Lorentzian metrics on an oriented four-dimensional manifold with Lagrangian density

$$(\mathcal{L}\omega)_{j^2_x g} = \bar{\pi}_2^*(R(g)\omega_g)_x,$$

where $g \in \Gamma(X_4, \mathcal{M})$ is a representative of the point $j^2_x g \in J^2(\mathcal{M}/X)$ and $R(g), \omega_g$ are the scalar curvature and volume element associated to g.

For each metric $g \in \Gamma(X_4, \mathcal{M})$, the differential operator P_g between the vector bundles TX_4 and $g^*V(\mathcal{M}) = S^2(T^*X_4)$ is given by

$$P_g(D) = -L_D g = -Sd^{\nabla_g}(i_D g), \quad D \in \mathfrak{X}(X), \tag{3.12}$$

where ∇_g is the Levi-Civita connection of g and S the symmetrization operator.

From (3.12), it follows that the lifting $D \mapsto \tilde{D}$ is vertically transitive, and also the following formula for the symbol of P_g:

$$\sigma(P_g)(D_x, \omega_x) = -S(\omega_x \otimes i_{D_x g}), \quad D_x \in T_x X_4. \tag{3.13}$$

Let $\mathcal{E} = \mathcal{E}^2 \otimes \omega_g \in \Gamma(X_4, S^2(T^*X_4) \otimes \bigwedge^4 T^*X_4)$. Let us denote by $\mathcal{E}_1^1$ and $\mathcal{E}_2$ the contraction of $\mathcal{E}^2$ with q and $g \otimes g$, respectively. By definition (3.3) of the adjoint operator (bearing in mind the classical formula $\mathrm{div}(\mathcal{E}_1^1(D)) = (\mathrm{div}_g \mathcal{E}_2)(D) + \mathrm{trace}(\mathcal{E}_1^1 \circ d^{\nabla_g} D)$), one obtains the following expression for P_g^+:

$$P_g^+(\mathcal{E}) = \mathrm{div}_g \mathcal{E}_2 \otimes \omega_g.$$

In particular, taking into account that the Euler–Lagrange operator $\mathcal{E} : g \in \Gamma(X_4, \mathcal{M}) \mapsto \mathcal{E}(g) = \mathcal{E}^2(g) \otimes \omega_g$ of the variational problem that we are considering is the Einstein tensor of the metric g (that is, $\mathcal{E}_2(g) = \mathrm{Eins}(g) = 2\,\mathrm{Ric}(g) - R(g)g$), the fundamental identity $P_s^+(\mathcal{E}(s)) = 0$ used in the proof of Theorem 3.1 (formula (3.9)) here takes the form

$$\mathrm{div}_g\, \mathrm{Eins}(g) = 0,$$

a well-known equality of Riemannian geometry that is known to be a direct consequence of Bianchi's second identity.

This observation is pertinent, because it allows us to interpret the equality $P_s^+(\mathcal{E}(s)) = 0$ as a generalized "Bianchi's identity" that must be fulfilled by the sections of the natural bundles we are dealing with.

Using the metric g, we can also express the stress–energy–momentum tensor, $\mathcal{T}(g)$; as an order 2 tensor in three different ways: as a mixed tensor $T_1^1(g)$ by the formula $i_{T_1^1(g)(D)}\omega_g = i_D \mathcal{T}(g)$, as a 2-covariant tensor $T_2(g)$, and as a 2-contravariant tensor $T^2(g)$, constructed by contractions of $T_1^1(g)$ with g and g^{-1}, respectively. We now have the following:

Proposition 3.3.

$$T_2(g) = \mathrm{Eins}(g).$$

Proof. From formula (3.10) (taken with respect to ω_g), it follows that for every pair of vector fields $D_1, D_2 \in \mathfrak{X}(X)$:

$$\begin{aligned} T_2(g)(D_1, D_2) &= T_1^1(g)(i_{D_1}g, D_2) = -\mathcal{E}^2(g)(\sigma(P_g)(D_2, i_{D_1}g)) \\ &= \mathcal{E}^2(g)[S(i_{D_1}g \otimes i_{D_2}g)] = \mathcal{E}^2(g)(i_{D_1}g, i_{D_2}g) \\ &= \mathcal{E}_2(g)(D_1, D_2) = \mathrm{Eins}(g)(D_1, D_2). \qquad \square \end{aligned}$$

Finally, taking the usual Poincaré–Cartan form Θ of second order variational problems (which in this case is known to be projectable to $J^1(Y/X)$), we can directly prove that (see Proposition 3.1 of [5]):

$$i_D \mathcal{T}(g) = \mu_\Theta(g)(D) + (d^{\nabla_g} D \cdot \omega_g),$$

which, as in Example 1, refines the fundamental equality (3.6) giving the term in the exact differential explicitly.

In particular, in this case this formula gives another proof of the main theorem, as well as a direct demonstration of the fact that the set of critical sections of the variational problem (Einstein metrics) coincides with the zero-level of the multimomentum map.

4. Metric parameterized Lagrangian densities

In this section we shall deal with the special important case of natural Lagrangian densities $\mathcal{L}\omega$ on the k-jet bundles $J^k(\mathcal{M} \times_X Y)$ of fibered products $\pi \times p : \mathcal{M} \times_X Y \to X$, where $\pi : \mathcal{M} \to X$ is a bundle of non-singular metrics of given signature on an n-dimensional oriented manifold X, $p : Y \to X$ an natural bundle with differential index ≤ 1 and where the natural lifting of vector fields from $\mathfrak{X}(X)$ to $\mathcal{M} \times_X Y$ has the form: $D \mapsto \tilde{D} = (\tilde{D}_{\mathcal{M}}, \tilde{D}_Y)$, where $D \mapsto \tilde{D}_{\mathcal{M}}$ and $D \mapsto \tilde{D}_Y$ are, respectively, the natural liftings from $\mathfrak{X}(X)$ to the bundle of metrics $\mathcal{M}$ and the natural bundle Y.

The Lagrangian density $\mathcal{L}\omega$ can be expressed in the form $L\omega_{\mathcal{M}}$, where $L \in C^\infty(J^k(\mathcal{M} \times_X Y))$ is a function, invariant by the natural action of Diff(X) over $J^k(\mathcal{M} \times_X Y)$, and $\omega_{\mathcal{M}}$ is the $\pi \times p$-horizontal n-form on $\mathcal{M} \times_X Y$ defined by the formula

$$(\omega_{\mathcal{M}})_{(g_x, y_x)}(\bar{D}^1, \ldots, \bar{D}^n) = \omega_g(\pi \times p)_* \bar{D}^1, \ldots, (\pi \times p)_* \bar{D}^n), \tag{4.1}$$

where ω_g is the volume element cononically associated to the oriented metric space $(T_x X, g_x)$.

The basic observation that justifies the denomination given to this section is the following: For each metric $g \in \Gamma(X, \mathcal{M})$, if we consider $(i_g)_{(k)} : J^k(Y/X) \to J^k(\mathcal{M} \times_X Y)$, the k-jet extension of the immersion of bundles over $X i_g : Y \to \mathcal{M} \times_X Y$ defined by $i_g(y) = (g(p(y)), y)$, $y \in Y$, then $(i_g)^*_{(k)}(L\omega_{\mathcal{M}}) = L_g\omega_g (L_g = (i_g)^*_{(k)} L \in C^\infty(J^k(Y/X)))$ defines a Lagrangian density on $J^k(Y/X)$.

We therefore have a family of variational problems $\{L_g\omega_g\}$ on $J^k(Y/X)$, g going through $\Gamma(X, \mathcal{M})$, which, by construction, are invariant by the natural action of the subgroups $\mathcal{G}(g) \subseteq$ Diff(X) of the isometries of g (or, infinitesimally, by the natural action of the real Lie algebras $\mathfrak{X}(g)$ of the Killing fields of g).

Definition 4.1. The stress–energy–momentum tensor of the variational problem $L_g\omega_g$ on $J^k(Y/X)$ is the correspondence that assigns to each section $s \in \Gamma(X, Y)$ the tensor $\mathcal{T}_g(s) = \mathcal{T}(g, s)$, which $\mathcal{T}(g, s)$ is the stress–energy–momentum tensor corresponding to the section $(g, s) \in \Gamma(X, \mathcal{M} \times_X Y)$ of the variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M} \times_X Y)$. Analogously, using ω_g, g and g^{-1}, $(T_g)^1_1$, $(T_g)_2$ and $(T_g)^2$ are defined from T^1_1 T_2 and T^2, respectively.

Remark 4.2. *If $s \in \Gamma(X, Y)$ is a critical section of the Lagrangian density $L_g\omega_g$, where g is a metric such that $(g, s) \in \Gamma(X, \mathcal{M} \times_X Y)$ is not critical for the Lagrangian density $L\omega_{\mathcal{M}}$, then the tensor $\mathcal{T}_g(s)$ does not necessarily vanish. This is the situation usually present in the applications. The restriction of the stress–energy–momentum tensor $\mathcal{T}_g$ to such critical sections therefore constitutes a family of non-trivial tensors.*

The question is now the following: How are the variational problems $L_g\omega_g$ on $J^k(Y/X)$ and $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M}\times_X Y)$ related by the immersion $i_g : Y \to \mathcal{M}\times_X Y$? In particular, how are the properties of the tensor $\mathcal{T}$ reflected on $\mathcal{T}_g$?

Let $(g,s)\in\Gamma(X,\mathcal{M}\times_X Y)$. By means of the identification of the vertical bundle $(g,s)^*V^*(\mathcal{M}\times_X Y) = S^2(TX)\oplus s^*V^*(Y)$, the Euler–Lagrange operator $\mathcal{E}$ of the variational problem $L\omega_{\mathcal{M}}$ can be decomposed in the form

$$\mathcal{E} : (g,s)\in\Gamma(X,\mathcal{M}\times_X Y)\mapsto \mathcal{E}(g,s) = [\mathcal{E}^2_{\mathcal{M}}(g,s),\mathcal{E}_Y(g,s)]\otimes\omega_g. \tag{4.2}$$

In these conditions, it is immediate from the local formula (2.4) that, for each metric $g\in\Gamma(X,\mathcal{M})$, the correspondence

$$(\mathcal{E}_Y)_g : s\in\Gamma(X,Y)\mapsto \mathcal{E}_Y(g,s)\otimes\omega_g \tag{4.3}$$

coincides with the Euler–Lagrange operator of the variational problem $L_g\omega_g$.

The Euler–Lagrange operator $(\mathcal{E}_Y)_g$ and $\mathcal{E}$ of both problems are thus i_g-related.

Regarding the respective families $\{\Theta\}_{L_g\omega_g}$ and $\{\Theta\}_{L\omega_{\mathcal{M}}}$ of Poincaré–Cartan forms, we have the following generalization to the morphism $i_g : Y\to\mathcal{M}\times_X Y$ of the functoriality property established in Section 2.

Proposition 4.3.

$$(i_g)^*_{(2k-1)} : \{\Theta\}_{L\omega_{\mathcal{M}}}\to\{\Theta\}_{L_g\omega_g}. \tag{4.4}$$

Proof. Let Θ be a Poincaré–Cartan form of the variational problem $L\omega_{\mathcal{M}}$ on $J^2(\mathcal{M}\times_X Y)$. By formulas (2.2) and the identifications

$$(VJ^l(\mathcal{M}\times_X Y))_{J^m} = (VJ^l(\mathcal{M})\oplus VJ^l(Y))_{J^m},\quad l\le m$$

(which also holds for the dual functor V^*), the differential forms Θ and $\mathrm{d}\Theta$ can be decomposed as follows:

$$\Theta = (\theta^k_{\mathcal{M}},\theta^k_Y)\bar{\wedge}(\Omega_{\mathcal{M}},\Omega_Y) + L\omega_{\mathcal{M}},$$
$$\mathrm{d}\Theta = (\theta^1_{\mathcal{M}},\theta^1_Y)\bar{\wedge}(\mathbb{E}_{\mathcal{M}},\mathbb{E}_Y) + (\theta^k_{\mathcal{M}},\theta^k_Y)\bar{\wedge}[(\theta^{k-1}_{\mathcal{M}},\theta^{k-1}_Y)\bar{\wedge}(\eta_{\mathcal{M}},\eta_Y)].$$

Applying $(i_g)^*_{(2k-1)}$ to these expressions, bearing in mind that, for every l, $(i_g)^*_{(l)}\theta^l_{\mathcal{M}} = 0$, and $(i_g)^*_{(l)}\theta^l_Y$ coincides with the structure 1-form of $J^l(Y/X)$ (which we will continue to denote θ^l_Y), we have

$$(i_g)^*_{(2k-1)}\Theta = \theta^k_Y\bar{\wedge}(i_g)^*_{(2k-1)}\Omega_Y + L_g\omega_g,$$
$$\mathrm{d}(i_g)^*_{(2k-1)}\Theta = \theta^1_Y\bar{\wedge}(i_g)^*_{(2k-1)}\mathbb{E}_Y + \theta^k_Y\bar{\wedge}(\theta^{k-1}_Y\bar{\wedge}(i_g)^*_{(2k-1)}\eta_Y),$$

which proves, again by (2.2), that $(i_g)^*_{(2k-1)}\Theta$ is a Poincaré–Cartan form of the variational problem $L_g\omega_g$ on $J^k(Y/X)$. □

Remark 4.4. *Analogously, it can be proved, by adapting the functoriality proof given in* [25] *to the morphism $i_g : Y\to\mathcal{M}\times_X Y$, that if Θ_∇ is the Poincaré–Cartan form of $L\omega_{\mathcal{M}}$*

corresponding to a linear connection ∇ on X, then $(i_g)^*_{(2k-1)}\Theta_\nabla$ is the Poincaré–Cartan form of $L_g\omega_g$ corresponding to the same connection. From here the bijection follows:

$$(i_g)^*_{(2k-1)} : \{\Theta_\nabla\}_{L\omega_{\mathcal{M}}} \xrightarrow{\sim} \{\Theta_\nabla\}_{L\omega_g\omega_g}. \tag{4.5}$$

In particular, if Θ_g is the Poincaré–Cartan form of the variational problem $L_g\omega_g$ on $J^k(Y/X)$, defined by the Levi-Civita connection ∇_g associated to g, then it holds that $\Theta_g = (i_g)^*_{(2k-1)}\Theta_{\nabla_g}$, where Θ_{∇_g} is the Poincaré–Cartan form of the variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M}\times_X Y)$ defined by the same connection.

In the case of $(k \le 2)$-order variational problems over manifolds of arbitrary dimensions or, for any k, over one-dimensional manifold, the former argument demonstrates that $(i_g)^*_{(2k-1)}$ transforms the corresponding canonical Poincaré–Cartan forms (2.6) into the other one, which on the other hand, can be immediately deduced from the local expressions for these forms simply by observing that the total derivatives $\mathbb{D}_j^\infty$ on $J^\infty(\mathcal{M}\times_X Y)$ and $(\mathbb{D}_Y)_j^\infty$ on $J^\infty(Y/X)$ are $(\pi_g)_\infty$-related.

The development followed in Example 2 of Section 3 (general relativity) can be extended to the variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M}\times_X Y)$ in the following way.

For each section $(g,s)\in\Gamma(X,\mathcal{M}\times_X Y)$, the differential operator $P_{(g,s)}$ between TX and $(g,s)^*V(\mathcal{M}\times_X Y) = S^2(TX)\otimes s^*V(Y)$ associated to the natural lifting $D\mapsto\tilde{D} = (\tilde{D}_{\mathcal{M}},\tilde{D}_Y)$ is given, on account of (3.12), by the formula

$$P_{(g,s)}(D) = [-Sd^{\nabla_g}(i_Dg), (P_Y)_s(D)], \quad D\in\mathfrak{X}(X), \tag{4.6}$$

where $(P_Y)_s$ is the differential operator between TX and $s^*V(Y)$ associated to the natural lifting $D\mapsto\tilde{D}_Y$ and to the section $s\in\Gamma(X,Y)$.

It follows from (4.6) and (3.13) that the symbol of $P_{(g,s)}$ is

$$\sigma(P_{(g,s)})(D_x,\omega_x) = [-S(\omega_x\otimes i_{D_x}g), \sigma(P_Y)_s(D_x,\omega_x)], \tag{4.7}$$

where $D_x\in T_xX$ and $\omega_x\in T_x^*X$.

Additionally, if $\mathcal{E} = (\mathcal{E}^2_{\mathcal{M}},\mathcal{E}_Y)\otimes\omega_g$ considered as a section of the bundle $[S^2(TX)\oplus s^*V^*(Y)]\otimes\bigwedge^n T^*X\to X$ then

$$P^+_{(g,s)}(\mathcal{E}) = \mathrm{div}_g(\mathcal{E}_{\mathcal{M}})_2\otimes\omega_g + (P_Y)^+_s(\mathcal{E}_Y\otimes\omega_g). \tag{4.8}$$

In particular, applying $P^+_{(g,s)}$ to the value $\mathcal{E}(g,s)$ of the Euler–Lagrange operator (4.2) of the variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M}\times_X Y)$, Bianchi's identity is obtained

$$(\mathrm{div}_g(\mathcal{E}_{\mathcal{M}})_2(g,s))\otimes\omega_g = -(P_Y)^+_s((\mathcal{E}_Y)_g(s)\otimes\omega_g), \tag{4.9}$$

where $(\mathcal{E}_Y)_g$ is the Euler–Lagrange operator (4.3) of the variational problem $L_g\omega_g$ on $J^k(Y/X)$. From these formulas, the stress–energy–momentum tensor $(T_g)^1_1$ of the variational problem $L_g\omega_g$ on $J^k(Y/X)$ can be calculated as follows:

Theorem 4.5.

$$(T_g)^1_1(s) = (\mathcal{E}_{\mathcal{M}})^1_1(g,s) - (\mathcal{E}_Y)_g(s)\cdot\sigma(P_Y)_s, \tag{4.10}$$

where $\sigma(P_Y)_s$ is considered as a 1-covariant, 1-contravariant tensor and the product "$\cdot$" is taken with respect to the duality between $s^*V^*(Y)$ and $s^*V(Y)$.

Proof. By definition (4.1) and formula (3.10) it follows that for every vector fields $D_1, D_2 \in \mathfrak{X}(X)$:

$$\begin{aligned}
&(T_g)_1^1(s)(i_{D_1}g, D_2)\\
&\quad = (T_1^1(g,s))(i_{D_1}g, D_2)\\
&\quad = -[\mathcal{E}^2_{\mathcal{M}}(g,s), (\mathcal{E}_Y)_g(s)][-S(i_{D_1}g \otimes i_{D_2}g), \sigma(P_Y)_s(i_{D_1}g, D_2)]\\
&\quad = \mathcal{E}^2_{\mathcal{M}}(g,s)(i_{D_1}g, i_{D_2}g) - (\mathcal{E}_Y)_g(s)(\sigma(P_Y)(i_{D_1}g, D_2))\\
&\quad = (\mathcal{E}_{\mathcal{M}})_1^1(g,s)(i_{D_1}g, D_2) - [(\mathcal{E}_Y)_g(s)\cdot\sigma(P_Y)_s](i_{D_1}g, D_2)\\
&\quad = [(\mathcal{E}_{\mathcal{M}})_1^1(g,s) - (\mathcal{E}_Y)_g(s)\cdot\sigma(P_Y)_s](i_{D_1}g, D_2). \qquad \square
\end{aligned}$$

Corollary 4.6 (Hilbert's formula). *If $s \in \Gamma(X,Y)$ is a critical section for the variational problem $L_g\omega_g$ on $J^k(Y/X)$, or if the natural lifting $D \mapsto \tilde{D}_Y$ has differential index 0, one has*

$$(T_g)_2(s) = (\mathcal{E}_{\mathcal{M}})_2(g,s) = \frac{\delta L(g,s)\omega_{\mathcal{M}}}{\delta g},$$

where the last term is intended with the classical notation. In particular, the tensor $(T_g)_2(s)$ is symmetric.

Proof. It suffices to apply formula (4.10), taking into account that in the first case $(\mathcal{E}_Y)_g(s) = 0$ and in the second one $\sigma(P_Y)_s = 0$. □

Corollary 4.7 (Divergence formula).

$$\begin{aligned}
(\mathrm{div}_g(T_g)_2(s)) \otimes \omega_g = {} & -(P_Y)_s^+[(\mathcal{E}_Y)_g(s) \otimes \omega_g]\\
& - \mathrm{div}_g[(\mathcal{E}_Y)_g(s)\cdot\sigma(P_Y)_s] \otimes \omega_g.
\end{aligned}$$

In particular, if $s \in \Gamma(X,Y)$ is a critical section of the variational problem $L_g\omega_g$ on $J^k(Y/X)$, the $\mathrm{div}_g(T_g)_2(s) = 0$.

Proof. It suffices to apply div_g to formula (4.10) bearing in mind Bianchi's identity (4.9). If $s \in \Gamma(X,Y)$ is a critical section of the variational problem $L_g\omega_g$ on $J^k(Y/X)$ then $(\mathcal{E}_Y)_g(s) = 0$, and this yields $\mathrm{div}_g(T_g)_2(s) = 0$. □

Finally, the fundamental formula (3.6) characterizing the stress–energy–momentum tensor $\mathcal{T}$, admits a translation to the tensors $\mathcal{T}_g$, proceeding as follows:

Lemma 4.8. *Let $D \in \mathfrak{X}(X)$. The necessary and sufficient condition for the natural liftings $\tilde{D}_Y$ and tildeD to be i_g-related by the immersion $i_g : Y \to \mathcal{M} \times_X Y$ is $D \in \mathfrak{X}(g)$.*

Proof. Let $y \in Y$ and $s \in \Gamma(X, Y)$ such that $s(x) = y$. Considering the isomorphism $\varphi S^2T^*X \oplus s^*V(Y) \xrightarrow{\sim} (g, s)^*V(\mathcal{M} \times_X Y)$, one has

$$\begin{aligned}\tilde{D}_{i_g(y)} &= \tilde{D}_{(g,s)(x)} = \mathrm{d}(g, s)_x D_x + (\tilde{D}^v_{(g,s)})_{(g,s)(x)} \\ &= \mathrm{d}(g, s)_x D_x + \varphi[(-L_D g, (\tilde{D}_Y)^v_s)]_{(g,s)(x)} \\ &= \mathrm{d}(g, s)_x D_x - [\varphi(L_D g)]_{(g,s)(x)} + [\varphi(\tilde{D}_Y)^v_s]_{(g,s)(x)} \\ &= (\mathrm{d}i_g)_y\{(\mathrm{d}s)_x D_x + [(\tilde{D}_Y)^v_s]_y\} - [\varphi(L_D g)]_{(g,s)(x)} \\ &= (\mathrm{d}i_g)_y(\tilde{D}_Y)_y - [\varphi(L_D g)]_{(g,s)(x)}\end{aligned}$$

so $(\mathrm{d}i_g)_y(\tilde{D}_Y)_y = (\tilde{D}_{i_g(y)})$ for every $y \in Y$ if and only if $L_D g = 0$. □

Theorem 4.9. *Let Θ be any Poincaré–Cartan form of the $\mathcal{G}(g)$-invariant variational problem $L_g\omega_g$ on $J^k(Y/X)$ such that $\Theta = (i_g)^*_{(2k-1)}\tilde{\Theta}$, where $\tilde{\Theta}$ is a Poincaré–Cartan form of the* Diff(X)*-invariant variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M} \times_X Y)$.*

If $\mu_\Theta : \Gamma(X, Y) \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(g), \Omega^{n-1})$ is the multimomentum map corresponding to the form Θ and the group of symmetries $\mathcal{G}(g)$, then for the stress–energy–momentum tensor $\mathcal{T}_g$ we have

$$i_D\mathcal{T}_g(s) = \mu_\Theta(s)(D) + \mathrm{d}\alpha, \quad s \in \Gamma(X, Y), \quad D \in \mathfrak{X}(g), \tag{4.11}$$

where α is an $(n-2)$-form on the manifold X.

Proof. Given $(g, s) \in \Gamma(X, \mathcal{M} \times_X Y)$, let us see that, for every $D \in \mathfrak{X}(g)$, $\mu_{\tilde{\Theta}}(g, s)(D) = \mu_\Theta(s)(D)$ holds. Indeed, given any point $x \in X$, by Lemma 4.8 and the equality $\Theta = (i_g)^*_{(2k-1)}\tilde{\Theta}$, one has

$$\begin{aligned}[\mu_{\tilde{\Theta}}(g, s)(D)]_x &= [(g, s)^*_{(2k-1)}(i_{\tilde{D}_{(2k-1)}}\tilde{\Theta})]_x = i_{((\tilde{D}_Y)_{(2k-1)})_{s_{(2k-1)}(x)}}\Theta \\ &= (s^*_{(2k-1)}i_{(\tilde{D}_Y)_{(2k-1)}}\Theta)_x = (\mu_\Theta(s)(D))_x.\end{aligned}$$

From here by Definition 4.1 and the fundamental formula (3.6), it follows that

$$i_D\mathcal{T}_g(s) = i_D\mathcal{T}(g, s) = \mu_{\tilde{\Theta}}(g, s)(D) + \mathrm{d}\alpha. \quad \square$$

Remark 4.10. *By the equality (4.11), for every $D \in \mathfrak{X}(g)$, the function $s \in \Gamma(X, Y) \mapsto s^*_{(2k-1)}(i_{\tilde{D}_{(2k-1)}}\Theta)$ defined on the set of sections $\Gamma(X, Y)$ by the Noether invariant $i_{\tilde{D}_{(2k-1)}}\Theta$ corresponding to the infinitesimal symmetry $\tilde{D}$ of the variational problem $L_g\omega_g$ on $J^k(Y/X)$ is given in terms of the stress–energy–momentum tensor $\mathcal{T}_g$ by the formula: $s \in \Gamma(X, Y) \mapsto i_D\mathcal{T}_g(s) - \mathrm{d}\alpha$.*

In particular, for $X = \mathbb{R}^4$ with the Minkowski metric g_0, applying the previous considerations to the translations $\partial/\partial x_i$ of the real Lie algebra $\mathfrak{X}(g_0)$ of the Poincaré–Group $\mathcal{G}(g_0)$, one has

$$T_i^j(s) = t_i^j(s) + \nabla_l K_i^{jl}. \tag{4.12}$$

where

$$i\partial/\partial x_i = T_i^j(s)\omega_i, \quad s^*[i(\tilde{\partial}/\partial x_i)_{(2k-1)}\Theta] = t_i^j(s)\omega_j$$

and

$$\alpha = K_i^{jl} i\partial/\partial x_l i\partial/\partial x_j\omega.$$

In the case of first order variational problems and taking the usual Poincaré–Cartan form for them, in the classical literature t_i^j is called the "canonical tensor" of the corresponding field theory, (4.12) being the formula that originally allowed Belinfante to obtain a "stress–energy–momentum tensor", T_i^j, "correcting" the canonical tensor t_i^j by means of an additive term $\nabla_l K_i^{jl}$ of "divergence type" (see [2]).

5. Examples

The two examples we offer in this section have a common characteristic. They are (≤ 2)-order variational problems parametrized by metrics defined by certain (≤ 1)-order constrained variational problems, see [5,9] for a general setup for this kind of problems.

Example 3: Electromagnetic field. As is well known, in its version with potentials, this theory is defined as a variational problem on $J^1(\mathcal{M} \times_{X_4} T^*X_4)$, where $\pi : \mathcal{M} \to X_4$ is the bundle of Lorentzian metrics on an oriented four-dimensional manifold X_4, $p : T^*X \to X_4$ is the bundle of "electromagnetic potentials", and the Lagrangian density is given by the formula

$$(\mathcal{L}\omega)_{j_x^1(g,A)} = \tfrac{1}{2}\|d_x A\|_g^2 \omega_{g_x}, \tag{5.1}$$

where $(g, A) \in \Gamma(X_4, \mathcal{M} \times_{X_4} T^*X)$ is a representative of the point $j_x^1(g, A)$, and $\| \ \|_g$ and ω_g are the norm and the volume element associated to g, respectively.

By the usual interpretation of the sections $F \in \Gamma(X_4, \bigwedge^2 T^*X)$ as "field intensities", we can consider this variational problem as the pull-back by the submersion

$$\varphi : j_x^1(g, A) \in J^1(\mathcal{M} \times_{X_4} T^*X) \mapsto (g_x, (\mathrm{d}A)_x) \in \mathcal{M} \times_{X_4} \bigwedge^2 T^*X$$

of the variational problem on $\mathcal{M} \times_{X_4} \bigwedge^2 T^*X$ defined by the Lagrangian density $(\bar{\mathcal{L}})(g_x, F_x) = 1/2\|F_x\|_g^2\omega_{g_x}$, where the sections (g, F) of the bundle $\mathcal{M} \times_{X_4} \bigwedge^2 T^*X \to X_4$ verify the constraint condition $\mathrm{d}F = 0$ (first group of Maxwell equations), and where the notion of "stationariness" is taken with respect to the constraint-preserving "variationa" $\delta_t g = g + tg'$, $\delta_t F = F + t\,\mathrm{d}A'$, g' and A' being an arbitrary metric and a 1-form on X_4, respectively.

Applying the results from Section 4 to the Lagrangian density (5.1), we have the following: Let $(g, A) \in \Gamma(X_4, \mathcal{M} \times_{X_4} T^*X)$. Using the identification $(g, A)^*V^*(\mathcal{M} \times_{X_4} T^*X) = S^2(TX_4) \oplus TX_4$, a straightforward local calculation proves that the Euler–Lagrange operator $\mathcal{E}$ of the variational problem (5.1) can be decomposed in the form: $\mathcal{E} : (g, A) \mapsto [\mathcal{E}_{\mathcal{M}}^2, \mathcal{E}_{T^*X_4}(g, A)] \oplus \omega_g$, where

$$\begin{aligned}\mathcal{E}^2_{\mathcal{M}}(g,A) &= (\tfrac{1}{4}g^{ij}g^{hk}g^{lm}F_{hl}F_{km} - F^i_l F^{jl})\,\mathrm{d}x_i\otimes \mathrm{d}x_j\\ &= \tfrac{1}{2}\|F\|^2_g g^{-1} - F^1_1\cdot F^2,\end{aligned}\tag{5.2}$$

$$\begin{aligned}\mathcal{E}_{T^*X_4}(g,A) &= -\frac{1}{\sqrt{-\det g}}\sum_j \frac{\partial}{\partial x_j}(\sqrt{-\det g}F^{ji})\,\mathrm{d}x_i\\ &\quad -\mathrm{div}_g F^2,\end{aligned}\tag{5.3}$$

with $F=\mathrm{d}A$.

By (4.3), for each metric g, the Euler–Lagrange operator defined by (5.1) on T^*X_4 is

$$(\mathcal{E}_{T^*X_4})_g : A\in\Gamma(X_4,T^*X_4)\mapsto(\mathrm{div}_g F^2)\otimes\omega_g.\tag{5.4}$$

On the other hand, the 2-forms F such that $\mathrm{d}F=0, \mathrm{div}_g F^2=0$ are the critical sections of the constrained variational problem on $\bigwedge^2 T^*X_4$ considered at the beginning of the example.

Thus, both variational problems (the free, first order one on the "bundle of potentials" T^*X_4 and the constrained zero order one on the "bundle of field intensities" $\bigwedge^2 T^*X_4$) are φ-related by the submersion

$$\varphi : j^1_x A\in J^1(T^*X_4)\to(\mathrm{d}A)_x\in\bigwedge^2 T^*X_4.$$

Let us now see the expression for the formulas (4.6)–(4.8) and for Bianchi's identity (4.9) in this case. In order to obtain them, it suffices to substitute in these formulas the respective expressions of $(P_{T^*X_4})_A$, $\sigma(P_{T^*X_4})_A$ and $(P_{T^*X_4})^+_A$. Applying the definitions in Section 3, one obtains

$$\begin{aligned}&(P_{T^*X_4})_A(D) = -L_D A,\quad D\in\mathfrak{X}(X_4),\\ &[\sigma(P_{T^*X_4})_A](D_x,\omega_x) = -A_x(D_x)\omega_x,\quad D_x\in T_xX_4,\ \omega_x\in T^*_xX_4,\\ &(P_{T^*X_4})^+_A(\mathcal{E}_{T^*X_4}\otimes\omega_g) = [i_{\mathcal{E}_{T^*X_4}}\mathrm{d}A + (\mathrm{div}_g\mathcal{E}_{T^*X_4})A]\otimes\omega_g,\end{aligned}$$

with $\mathcal{E}_{T^*X_4}\otimes\omega_g\in\Gamma(X_4,TX_4\otimes\bigwedge^4 T^*X_4)$.

In particular, bearing in mind (5.3), formula (4.9) for Bianchi's identity in this case takes the form

$$\mathrm{div}_g(\mathcal{E}_{\mathcal{M}})_2(g,A) = i_{\mathrm{div}_g F}F,\tag{5.5}$$

where $(\mathcal{E}_M)_2(g,A)$ is the 2-covariant expression of the tensor defined by the formula (5.2).

For the 2-covariant expression of the stress–energy–momentum tensor, Theorem 4.5 gives

$$(T_g)_2(A) = (\mathcal{E}_{\mathcal{M}})_2(g,A) - A\otimes\mathrm{div}_g F.$$

This tensor therefore coincides with the one defined by the Hilbert formula (Corollary 4.6) only over the critical sections (that is, when $\mathrm{div}_g F=0$). In this case, one obtains the well-known formula

$$(T_g)_2(A) - (\mathcal{E}_{\mathcal{M}})_2(g,A) = \tfrac{1}{2}\|F\|^2_g g - F^1_1\cdot F_2,$$

which is solely expressed in terms of the field intensities.

Regarding the divergence formula (Corollary 4.6), this takes the form

$$\mathrm{div}_g(T_g)_2(A) = i_{\mathrm{div}_g F} F - \mathrm{div}_g(A \otimes \mathrm{div}\,_g F).$$

Finally, since this is a first order variational problem, on applying the results of Example 1 to the corresponding momentum $(n-1)$-form Ω, the corrective Belinfante–Rosenfed term of formula (4.11) takes the form

$$\mathrm{d}\alpha = \mathrm{d}[\Omega(\sigma(P_{T^*X_4})_A(D))] = -\mathrm{d}(A(D)F^2 \cdot \omega_g), \quad D \in \mathfrak{X}(g),$$

where "·" denotes the contraction of the two indices of F^2 with the volume element ω_g.

Example 4: Non-perfect relativistic fluids. In this example, the bundle T^*X_4 of Example 3 is replaced by the bundle $Y = X_4 \times M_3$, the direct product of X_4 with a three-dimensional manifold, oriented by a volume element η_3 maintaining the bundle $\pi : \mathcal{M} \to X_4$ of Lorentzian metrics on X_4 as the parameterizing space.

In fact, this is the starting point of the variational theory of relativistic perfect fluids by Kijowski et al. [17], which is based on the following construction:

Let $\varphi : J^1(\mathcal{M} \times_{X_4} Y) \to \mathcal{M} \times_{X_4} TX_4$ be the morphism of natural bundles over X_4 defined by the formula

$$\varphi : j^1_x(g, f) \in J^1(\mathcal{M} \times_{X_4} Y) \mapsto [g_x, i_{*_g(f^*\eta_3)_x} g^{-1}] \in \mathcal{M} \times_{X_4} TX_4, \tag{5.6}$$

where $f : X_4 \to M_3$ on the right-hand side must be understood using the natural identification $\Gamma(X_4, Y) = \mathrm{Map}(X_4, M_3)$ and where $*_g$ is the Hodge operator with respect to the metric g.

Considering the open subbundle of $\mathcal{M} \times_{X_4} TX_4$ defined by the pairs (g_x, D_x), where D_x is a time-like, future-ponting tangent vector with respect to the metric g_x, and the subbundle of $J^1(\mathcal{M} \times_{X_4} Y)$ obtained by applying φ^{-1} to the previous subbundle (for simplicity we shall continue to denote them as their respective ambient bundles), the Lagrangian density of the perfect relativistic fluids is defined by $\varphi^*(L\omega_{\mathcal{M}})$, where $L : \mathcal{M} \times_{X_4} TX_4 \to \mathbb{R}$ is a function invariant by the natural action of Diff(X_4) on $\mathcal{M} \times_{X_4} TX_4$. That is

$$L : (g_x, D_x) \in \mathcal{M} \times_{X_4} TX_4 \mapsto \mathcal{L}\left(\sqrt{-\|D_x\|^2_{gx}}\right),$$

with $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}$ an arbitrary differentiable function.

The variational problem defined by $\varphi^*(L\omega_{\mathcal{M}})$ on $J^1(\mathcal{M} \times_{X_4} Y)$ constitutes the "Lagrangian" formulation of the relativistic perfect fluids proposed in [17], while the "Eulerian" version of these fluids is given by the variational problem on $\mathcal{M} \times_{X_4} TX_4$ of Lagrangian density $L\omega_{\mathcal{M}}$, where the sections $(g, D) \in \Gamma(X_4, \mathcal{M} \times_{X_4} TX_4)$ fulfill the constraint condition $\mathrm{div}_g D = 0$ (continuity equation) and where the notion of "stationariness" is taken with respect to the "variations" preserving the constraint

$$\delta_{tg} = g + tg', \quad \delta_t D = D + t\{[D, D'] - (\mathrm{div}_g D' + g^{-1} \cdot g')D\},$$

with g' and D' being an arbitrary metric and a vector field on X_4, respectively.

The analogy with Example 3 is evident: D represents a fluid on the Lorentzian manifold (X_4, g) of mass density $\rho = \sqrt{-\|D\|_g^2}$ and field of velocities $U = D/\rho$, while the functions $f : X_4 \to M_3$ are interpreted as a special kind of "hydrodynamic potentials" that solve the continuity equation (note that from $d\eta_3 = 0$ follows the equality $\operatorname{div}(*_g(f^*\eta_3)) = 0$).

Remark 5.1. *The former analogy becomes closer if we take as the space of "Eulerian variables"* $\mathcal{M} \times_{X_4} \bigwedge^3 T^*X_4$ *instead of* $\mathcal{M} \times_{X_4} TX_4$, *whereby the continuity equation transforms into* $dF_3 = 0$ *on the sections* $(g, F_3) \in \Gamma(X_4, \mathcal{M} \times_{X_4} \bigwedge^3 T^*X_4)$. *The relation with the standard formulation is established by the vector bundle isomorphism*

$$(g_x, D_x) \in \mathcal{M} \times_{X_4} TX_4 \mapsto (g_x, i_{D_x}\omega_x) \in \mathcal{M} \times_{X_4} \bigwedge^3 T^*X_4.$$

The morphism that relates this formulation with the potential theory is

$$j_x^1(g, f) \in J^1(\mathcal{M} \times_{X_4} Y) \mapsto (g_x, (F^*\eta_3)_x) \in \mathcal{M} \times_{X_4} \bigwedge^3 T^*X_4,$$

which, for the computations, is much simpler than the one defined by formula (5.6) *(this new setting can be found in* [5]*).*

If we consider the natural prolongation $\bar{\varphi} : J^2(\mathcal{M} \times_{X_4} Y) \to J^1(\mathcal{M} \times_{X_4} TX_4)$ of the morphism (5.6) defined by the formula

$$\bar{\varphi} : j_x^2(g, f) \in J^2(\mathcal{M} \times_{X_4} Y) \mapsto j_x^1(\varphi \circ j^1(g, f)) \in J^1(\mathcal{M} \times_{X_4} TX_4) \qquad (5.7)$$

The previous theory can be generalized, taking as Lagrangian an arbitrary function $L : J^1(\mathcal{M} \times_{X_4} TX_4) \to \mathbb{R}$, invariant by the natural action of Diff(X_4) on $J^1(\mathcal{M} \times_{X_4} TX_4)$.

The variational problem defined by $\bar{\varphi}(L\omega_{\mathcal{M}})$ on $J^2(\mathcal{M} \times_{X_4} TX_4)$ would be the "Lagrangian" version of a theory of "non-perfect relativistic fluids", whose "Eulerian" formulation would be given by the variational problem on $J^1(\mathcal{M} \times_{X_4} TX_4)$ of Lagrangian density $L\omega_{\mathcal{M}}$ with the same constraint and notion of stationariness for the sections $(g, D) \in \Gamma(X_4, \mathcal{M} \times_{X_4} TX_4)$ considered for the perfect fluids.

Applying the general theory to this case we will have the following:

Let $(g, f) \in \Gamma(X_4, \mathcal{M} \times_{X_4} Y)$. Making use of the identification $(g, f)^*V^*(\mathcal{M} \times_{X_4} Y) = S^2TX_4 \bigoplus f^*T^*M_3$, the Euler–Lagrange operator $\mathcal{E}$ of the variational problem on $J^2(\mathcal{M} \times_{X_4} Y)$ of Lagrangian density $\bar{\varphi}^*(L\omega_{\mathcal{M}})$ decomposes in the form $\mathcal{E} : (g, f) \mapsto [\mathcal{E}_{\mathcal{M}}^2(g, f), \mathcal{E}_Y(g, f)] \otimes \omega_g$.

Analogously, given a section $(g, D) \in \Gamma(X_4, \mathcal{M} \times_{X_4} TX_4)$, the identification $(g, f)^*V^*(\mathcal{M} \times_{X_4} TX_4) = S^2TX_4 \bigotimes T^*X_4$ allows us to decompose the Euler–Lagrange operator $\hat{\mathcal{E}}$ of the variational problem on $J^1(\mathcal{M} \times_{X_4} TX_4)$ of Lagrangian density $L\omega_{\mathcal{M}}$ (considered as an unconstrained problem) in the form

$$\hat{\mathcal{E}} : (g, D) \mapsto [\hat{\mathcal{E}}_{\mathcal{M}}^2(g, D), \hat{\mathcal{E}}_{TX_4}(g, D)] \otimes \omega_g.$$

With these conditions, a fundamental result holds:

Proposition 5.2 [5].

$$\mathcal{E}^2_{\mathcal{M}}(g, f) = \hat{\mathcal{E}}^2_{\mathcal{M}}(g, D) - [i_D \hat{\mathcal{E}}_{TX_4}(g, D)]g^{-1}, \tag{5.8}$$

$$f^* \mathcal{E}_Y(g, f) = i_D \, d\hat{\mathcal{E}}_{TX_4}(g, D), \tag{5.9}$$

where $D = i_{*_g(f^*\eta_3)} g^{-1}$.

Proof. The proof given in [5] is very simple: it is based on a comparison of the formulas of variation for the free variational problem $\bar{\varphi}^*(L\omega_{\mathcal{M}})$ on $J^2(\mathcal{M} \times_{X_4} Y)$ and the constrained variational problem $L\omega_{\mathcal{M}}$ on $J^1(\mathcal{M} \times_{X_4} TX_4)$ on two sections (g, f) and $(g, D = i_{*_g(f^*\eta_3)} g^{-1})$ of each bundle, related by the morphism $\bar{\varphi}$.

Another way to demonstrate these formulas, without using variational calculus with constraints, is simply by local computations. Let us sketch this procedure.

Let (x_i), $i = 1, \ldots, 4$ be a local coordinate system on X_4, (x_i, g_{ij}, y^i) the coordinates induced on $\mathcal{M} \times_{X_4} TX_4$, and (z^α), $\alpha = 1, 2, 3$ unimodular local coordinates on M_3 with respect to η_3. If $g_{ij} = g_{ij}(x)$, $z^\alpha = z^\alpha(x)$ are the equations of a section $(g, f) \in \Gamma(X_4, \mathcal{M} \times_{X_4} Y)$, then the equations of $\varphi(j^1(g, f)) = (g, i_{*_g(F^*\eta_3)} g^{-1}) \in \Gamma(X_4, \mathcal{M} \times_{X_4} TX_4)$ are $g_{ij} = g_{ij}(x)$, $y^h = y^h(x)$, with

$$y^h(x) = \frac{1}{\sqrt{-\det g}} (-1)^{h-1} \underset{\hat{h}}{\begin{vmatrix} z^1_1(j^1_x\varphi) & \ldots & z^1_4(j^1_x\varphi) \\ \vdots & \vdots & \vdots \\ z^3_1(j^1_x\varphi) & \ldots & z^3_4(j^1_x\varphi) \end{vmatrix}}, \tag{5.10}$$

where $\hat{h}$ denotes that the hth column is omitted and $z^i_h(x) = \partial z^i(x)/\partial x_h$.

If we derive $y^h(x)$ with respect to x_k, we have

$$\frac{\partial y^h(x)}{\partial x_k} = -g^{st}(x) y^h(x) \frac{\partial g^{st}(x)}{\partial x_k} + \frac{1}{\sqrt{-\det g}} (-1)^{h-1} \frac{\partial}{\partial x_k} \underset{\hat{h}}{\begin{vmatrix} z^1_1(j^1_x\varphi) & \ldots & z^1_4(j^1_x\varphi) \\ \vdots & \vdots & \vdots \\ z^3_1(j^1_x\varphi) & \ldots & z^3_4(j^1_x\varphi) \end{vmatrix}} \tag{5.11}$$

Formulas (5.10) and (5.11) give the equations of the morphism $\bar{\varphi}$ simply by substitution of the “derivatives” appearing in them by “jet coordinates” (that is, $\partial g_{st}/\partial x_k \mapsto g^{st}_k$, $\partial y^h/\partial x_k \mapsto y^h_k$, $\partial z^\alpha/\partial x_k \mapsto z^\alpha_k$ and $\partial^2 z^\alpha/(\partial x_k \partial x_s) \mapsto z^\alpha_{(k,s)}$).

If $\mathcal{L}(g_{ij}, y^h, g^{st}_k, y^h_k)\, dx_1 \wedge \cdots \wedge dx_4$ is the local expression of $L\omega_{\mathcal{M}}$, these expressions allow us to explicitly compute the left hand side of (5.8) and (5.9) merely by applying the chain rule to the corresponding expressions obtained for $\bar{\varphi}^*(L\omega_{\mathcal{M}}) = (\bar{\varphi}^*\mathcal{L})\, dx_1 \wedge \cdots \wedge dx_4$.

For $\mathcal{E}^2_{\mathcal{M}}(g, f) = \sum_{i\le j} \mathcal{E}^{ij}_{\mathcal{M}}(\partial/\partial x_i)\cdot(\partial/\partial x_j)$, we shall have

$$\sqrt{-\det g}\,\mathcal{E}^{ij}_{\mathcal{M}} = \frac{\partial\bar\varphi^*\mathcal{L}}{\partial g_{ij}} - \frac{\partial}{\partial x_l}\left(\frac{\partial\bar\varphi^*\mathcal{L}}{\partial g^{ij}_l}\right)$$

$$= \frac{\partial\mathcal{L}}{\partial g_{ij}} + \frac{\partial\mathcal{L}}{\partial y^h}\frac{\partial y^h}{\partial g_{ij}} + \frac{\partial\mathcal{L}}{\partial y^h_k}\frac{\partial y^h_k}{\partial g_{ij}} - \frac{\partial}{\partial x_l}\left(\frac{\partial\mathcal{L}}{\partial g^{ij}_l} + \frac{\partial\mathcal{L}}{\partial y^h_k}\frac{\partial y^h_k}{\partial g^{ij}_l}\right).$$

Computing the derivative of y^h with respect to g_{ij} in (5.10), one has $\partial y^h/\partial g_{ij} = -g^{ij}y^h$, and analogously, the derivatives of y^h_k with respect to g_{ij} and g^{ij}_l in (5.11) are

$$\frac{\partial y^h_k}{\partial g_{ij}} = -\frac{\partial}{\partial x_k}(g^{ij}y^h), \qquad \frac{\partial y^h_k}{\partial g^{ij}_l} = -g^{ij}y^h\delta^k_l.$$

Hence, substituting in $\sqrt{-\det g}\,\mathcal{E}^{ij}_{\mathcal{M}}$, we shall have

$$\sqrt{-\det g}\,\mathcal{E}^{ij}_{\mathcal{M}} = \frac{\partial\mathcal{L}}{\partial g_{ij}} - g^{ij}y^h\frac{\partial\mathcal{L}}{\partial y^h} - \frac{\partial\mathcal{L}}{\partial y^h_k}\frac{\partial}{\partial x_k}(g^{ij}y^h) - \frac{\partial}{\partial x_l}\left(\frac{\partial\mathcal{L}}{\partial g^{ij}_l}\right)$$

$$+ \frac{\partial}{\partial x_l}\left(\frac{\partial\mathcal{L}}{\partial y^h_k}\right)g^{ij}y^h + \frac{\partial\mathcal{L}}{\partial y^h_l}\frac{\partial}{\partial x_l}(g^{ij}y^h)$$

$$= \sqrt{-\det g}\,((\hat{\mathcal{E}}_{\mathcal{M}})_{ij} - [i_D\hat{\mathcal{E}}_{TX_4}(g, D)]g^{ij}),$$

which proves the equality (5.8) of the proposition.

Making a similar computation (longer, as more successive derivatives appear), one obtains the proof of (5.9). □

Remark 5.3. *For each metric g, the Lagrangian densities $\bar\varphi^*(L\omega_{\mathcal{M}})$ and $L\omega_{\mathcal{M}}$ define on $J^2(Y)$ and $J^1(TX_4)$ a (free) second order variational problem and a (constrained) first order variational problem, respectively. In* [5] *it is proved that the equations of the critical sections for the second problem are:* $\mathrm{div}_g D = 0$, $i_D\,\mathrm{d}\hat{\mathcal{E}}_{TX_4}(g, D) = 0$. *Therefore, by proposition 5.2, both variational problems are $\bar\varphi_g$-related by the morphism $\bar\varphi_g : f \in \Gamma(X_4, Y) \mapsto D = i_{[*_g(f^*\eta_3)]}g^{-1}$.*

In the case of the perfect fluids (that is, when the Lagrangian density is $L\omega_{\mathcal{M}} = \mathcal{L}(\rho)\sqrt{-\det g}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_4$), formulas (5.8) and (5.9) give the classical expressions of this theory. Indeed

$$\hat{\mathcal{E}}_{TX_4}(g, D) = \sum_j \hat{\mathcal{E}}_j\,\mathrm{d}x_j = \sum_j \frac{1}{\sqrt{-\det g}}\frac{\partial(\mathcal{L}(\rho)\sqrt{-\det g})}{\partial y^j}\,\mathrm{d}x_j$$

$$= \frac{\mathcal{L}'(\rho)}{\rho}i_D g,$$

$$
\begin{aligned}
\hat{\mathcal{E}}^2_{\mathcal{M}}(g, D) &= \sum_{i\le j} \hat{\mathcal{E}}^{ij} \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} \\
&= \sum_{i\le j} \frac{1}{\sqrt{-\det g}} \frac{\partial(\mathcal{L}(\rho)\sqrt{-\det g})}{\partial g_{ij}} \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} \\
&= \sum_{i\le j} \left(\frac{\partial \mathcal{L}(\rho)}{\partial g_{ij}} + \frac{1}{\sqrt{-\det g}} \mathcal{L}(\rho) \frac{\partial \sqrt{-\det g}}{\partial g_{ij}} \right) \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} \\
&= \sum_{i\le j} \left(\mathcal{L}'(\rho) \frac{\partial \rho}{\partial g_{ij}} + \mathcal{L}(\rho) g^{ij} \right) \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} \\
&= \sum_{i\le j} \left(-\frac{\mathcal{L}'(\rho)}{\rho} y^i y^j + \mathcal{L}(\rho) g^{ij} \right) \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} \\
&= -\frac{\mathcal{L}'(\rho)}{\rho} D \otimes D + \mathcal{L}(\rho) g^{-1}.
\end{aligned}
$$

Hence, substituting in (5.8) and (5.9), one obtains, respectively:

$$
\begin{aligned}
\mathcal{E}^2_{\mathcal{M}}(g, f) &= \frac{-\mathcal{L}'(\rho)}{\rho} D \otimes D + (\mathcal{L}(\rho) - \rho \mathcal{L}'(\rho)) g^{-1}, \\
f^* \mathcal{E}_Y(g, f) &= i_D \, \mathrm{d} \left[\frac{\mathcal{L}'(\rho)}{\rho} i_D g \right].
\end{aligned}
$$

In particular, the Eulerian expression of the second group of field equations for the perfect fluid is

$$
i_D \, \mathrm{d} \left[\frac{\mathcal{L}'(\rho)}{\rho} i_D g \right] = 0, \tag{5.12}
$$

which, setting $\mathcal{L}(\rho) = -\rho(1 + \epsilon(\rho))$ as usual and defining $p = \rho^2(\mathrm{d}(\epsilon(\rho))/\mathrm{d}\rho)$ and $\mu(\rho) = -\mathcal{L}(\rho)$, takes the well-known form

$$
U(p)\omega_U + \mathrm{d}p + (\mu + p) U^{\nabla} \omega_U = 0, \tag{5.13}
$$

where $U = D/\rho$, $\omega_U = -i_U g$ and ∇ is the Levi-Civita connection of g.

Returning to the general case, since the lifting $D \mapsto \tilde{D}_Y$ of vector fields of $\mathfrak{X}(X_4)$ to the bundle $Y = X_4 \times M_3$ is the trivial one, then for every $f : X_4 \to M_3$ the operators $(P_Y)_f$ and $(P_Y)^+_f$ are homomorphisms of vector bundles, and hence $\sigma(P_Y)_f = 0$. More explicitly, one has

$$
\begin{aligned}
&[(P_Y)_f(D)]_x = (\mathrm{d}f)_x D_x, \quad D \in \mathfrak{X}(X_4), \quad x \in X_4, \\
&\sigma(P_Y)_f(D_x, \omega_x) = 0, \quad D_x \in T_x X_4, \quad \omega_x \in T^*_x X_4, \\
&(P_Y)^+_f(\mathcal{E}_Y \otimes \omega_g)_x = (\mathcal{E}_Y)_x \otimes (\omega_g)_x, \\
&\mathcal{E}_Y \otimes \omega_g \in \Gamma\left(X_4, f^* T^* M_3 \otimes \bigwedge^4 T^* X_4 \right).
\end{aligned}
$$

In particular, bearing in mind (5.9), formula (4.9) for Bianchi's identity in this case takes the form

$$\mathrm{div}_g(\mathcal{E}_{\mathcal{M}})_2(g, f) = -i_D\, \mathrm{d}\hat{\mathcal{E}}_{TX_4}(g, D),$$

where $(\mathcal{E}_{\mathcal{M}})_2(g, f)$ is the 2-covariant expression of the tensor given by formula (5.8).

On the other hand, since $\sigma(P_Y) = 0$, Hilbert's formula (Corollary 4.6) yields

$$(T_g)_2(f) = (\mathcal{E}_{\mathcal{M}})_2(g, f) = (\hat{\mathcal{E}}_{\mathcal{M}})_2(g, D) - [i_D\hat{\mathcal{E}}_{TX_4}(g, D)]g. \tag{5.14}$$

In particular, for perfect fluids one has the well-known expression for their stress–energy–momentum tensor

$$(T_g)_2(f) = \frac{-\mathcal{L}'(\rho)}{\rho}\omega_D \otimes \omega_D + [\mathcal{L}(\rho) - \rho\mathcal{L}'(\rho)]g. \tag{5.15}$$

Regarding the divergence formula (Corollary 4.6), again by $\sigma(P_Y) = 0$, one obtains

$$\mathrm{div}_g(T_g)_2(f) = -i_D\, \mathrm{d}\hat{\mathcal{E}}_{TX_4}(g, D). \tag{5.16}$$

In particular, this equation implies, by (5.9), that for the submersions $f : X_4 \to \mathbb{R}^3$, the critical sections of the variational problem defined by $\bar{\varphi}^*(L\omega_{\mathcal{M}})$ on $J^2(Y)$ fixing the metrix, are characterized by the equation $\mathrm{div}_g(T_g)_2(f) = 0$.

As a second application of the general results just obtained, we shall address the natural Lagrangian density $L\omega_{\mathcal{M}}$ on $J^1(\mathcal{M} \times_{X_4} TX_4)$ given by

$$L(j_x^1(g, D)) = \tfrac{1}{2} \parallel \mathrm{d}\omega_D \parallel_g^2 (x) + \mathcal{L}\left(\sqrt{- \parallel \omega_D \parallel_g^2}\right)(x), \tag{5.17}$$

where $(g, D) \in \Gamma(X_4, \mathcal{M} \times X_4TX_4)$ is a representative of the point $j_x^1(g, D)$ and $\omega_D = i_Dg$.

This is the so-called non-linear vectorial Klein–Gordon Lagrangian on a Lorentzian manifold (X_4, g), whose field equations (as a free variational problem) are $\delta_g\, \mathrm{d}\omega_D - (\mathcal{L}'(\rho)/\rho)\omega_D = 0$ as can be easily seen by the local expression for the Lagrangian (5.17). Hence, taking into account the constraint equation $\mathrm{div}_g D = 0$, we obtain

$$\hat{\mathcal{E}}_{TX_4}(g, D) = \Box_g\omega_D - \frac{\mathcal{L}'(\rho)}{\rho}\omega_D, \tag{5.18}$$

where $\Box_g$ is the Laplacian for the given metric g.

Analogously, making use of the local expression of the Lagrangian (5.17) and deriving with respect to g_{ij} and g_l^{ij}, one obtains

$$(\hat{\mathcal{E}}_{\mathcal{M}})_2(g, D) = Lg - \mathrm{d}\omega_D \cdot \mathrm{d}\omega_D + S(\Box_g\omega_D \otimes \omega_D) - \frac{\mathcal{L}'(\rho)}{\rho}\omega_D \otimes \omega_D, \tag{5.19}$$

where S is the symmetrization operator and $\mathrm{d}\omega_D \cdot \mathrm{d}\omega_D$ denotes the contraction of the first index of $\mathrm{d}\omega_D$ with the contravariant index of the (1, 1)-tensor defined by $\mathrm{d}\omega_D$ and the metric.

Substituting (5.18) and (5.19) in (5.8) and (5.9), we obtain, respectively

$$(\mathcal{E}_{\mathcal{M}})_2(g, f) = Lg - \mathrm{d}\omega_D \cdot \mathrm{d}\omega_D + S(\Box_g \omega_D \otimes \omega_D) - \frac{\mathcal{L}'(\rho)}{\rho} \omega_D \otimes \omega_D - (\Box_g \omega_D)(D) g - \rho \mathcal{L}'(\rho) g,$$

$$f^* \mathcal{E}_Y(g, f) = i_D \,\mathrm{d} \left(\Box_g \omega_D - \frac{\mathcal{L}'(\rho)}{\rho} \omega_D \right).$$

In particular, the Eulerian expression of the second group of field equations for these fluids are

$$i_D \,\mathrm{d} \Box_g \omega_D - i_D \,\mathrm{d} \left(\frac{\mathcal{L}'(\rho)}{\rho} \omega_D \right) = 0,$$

which as can be seen, has a second term the Euler equation (5.12) for the perfect fluids corresponding to the Lagrnagian density $\mathcal{L}(\rho)\omega_{\mathcal{M}}$ defined by the second term of the Lagrangian (5.16) under consideration.

Additionally, substituting in (5.14), we obtain the following expression for the stress–energy–momentum tensor

$$(T_g)_2(f) = \left[\frac{1}{2} \parallel \mathrm{d}\omega_D \parallel_g^2 - (\Box_g \omega_D)(D) \right] g + S(\Box_g \omega_D \otimes \omega_D) - \mathrm{d}\omega_D \cdot \mathrm{d}\omega_D + \left\{ -\frac{\mathcal{L}'(\rho)}{\rho} \omega_D \otimes \omega_D + [\mathcal{L}(\rho) - \rho \mathcal{L}'(\rho)] g \right\},$$

whose last term $\{\cdots\}$ coincides, as would be expected, with the stress–energy–momentum tensor (5.15) for the perfect fluids with Lagrangian density $\mathcal{L}(\rho)\omega_{\mathcal{M}}$.

Apart from its possible practical interest, the present case is not only geometrically very appealing, but also allows us to set the old Klein–Gordon theory within the general framework of relativistic hydrodynamics. In particular, the continuity equation is no more than the hydrodynamical translation of the well-known Lorentz condition. On the other hand, the solutions of the Klein–Gordon equations ($\delta_g \omega_D = 0$, $\Box_g \omega_D - (\mathcal{L}'(\rho)/\rho)\omega_D = 0$) define a subclass of fluids with a stress–energy–momentum tensor that coincides with the one that would correspond to the Klein–Gordon Lagrangian as a first order free variational problem.

6. Minimal gravitational interactions

Example 3 (general relativity) can be generalized in an obvious way to a Lagrangian density $L_{\mathcal{M}}\omega_{\mathcal{M}}$ on $J^k(\mathcal{M})$, where $\pi : \mathcal{M} \to X$ is a bundle of non-singular metrics of a given signature on an oriented manifold X and $L_{\mathcal{M}} \in C^\infty(J^k(\mathcal{M}))$ an arbitrary function, invariant by the natural action of Diff(X). In particular, if we denote by $\mathrm{Eins}(L_{\mathcal{M}})$ the corresponding Euler–Lagrange operator, it will satisfy the generalized Bianchi's identity $\mathrm{div}_g \,\mathrm{Eins}(L_{\mathcal{M}}) = 0$. Analogously, the stress–energy–momentum tensor of the natural variational problem $L_{\mathcal{M}}\omega_{\mathcal{M}}$ on $J^k(\mathcal{M})$ coincides with $\mathrm{Eins}(L_{\mathcal{M}})$.

Let $L\omega_{\mathcal{M}}$ and $L_{\mathcal{M}}\omega_{\mathcal{M}}$ be natural Lagrangian densities on the jet bundles $J^k(\mathcal{M} \times_X Y)$ and $J^k(\mathcal{M})$, whose variational problems will be called "source field" and "gravitational field", respectively.

Definition 6.1. A "minimal gravitational interaction" between the source and gravitational fields is the natural variational problem on $J^k(\mathcal{M} \times_X Y)$ with Lagrangian density $(L + L_{\mathcal{M}})\omega_M$.

As is known, this definition constitutes the formalization of the classical Einstein trick, following which: "a field, invariant by the Poincaré group, can be converted into another, invariant by the whole space–time diffeomorphisms, in order to satisfy the general relativity principle". More precisely: given a variational problem on $J^k(Y)$ with Lagrangian density $L_g\omega_g$ invariant by the subgroup $\mathcal{G}(g) \subset \mathrm{Diff}(X)$ of the isometries of g, we can obtain a Diff (X)-invariant variational problem taking as Lagrangian density $(L + L_{\mathcal{M}})\omega_M$ on $J^k(\mathcal{M} \times_X Y)$, where $(L\omega_M)(g) = L_g\omega_g$ and $L_{\mathcal{M}} \in C^\infty(J^k(\mathcal{M}))$ is a Diff(X)-invariant function. In Einstein's theory: $k = 2$, $\mathcal{M}$ = Lorentzian metrics bundle on X_4 and $L_{\mathcal{M}} = R(g)$, the scalar curvature of g.

Definition 6.2. The stress–energy–momentum tensor of the minimal gravitational interaction $(L + L_{\mathcal{M}})\omega_{\mathcal{M}}$ is the stress–energy–momentum tensor of the source field $L\omega_{\mathcal{M}}$.

Hence, Theorem 4.5 continues to give the explicit expression of this tensor, which, bearing in mind that the Euler–Lagrange operator of the Lagrangian density $(L + L_{\mathcal{M}})\omega_{\mathcal{M}}$ is

$$\mathcal{E} : (g, s) \mapsto [(\mathrm{Eins}(L_{\mathcal{M}}))(g) + \mathcal{E}_{\mathcal{M}}(L)(g, s), (\mathcal{E}_Y(L))_g(s)] \otimes \omega_g$$

allows us to characterize the critical sections of the interaction as the solutions of the system of equations

$$\mathrm{Eins}(L_{\mathcal{M}})(g) = (T_g)_2(s), \qquad (\mathcal{E}_Y(L))_g(s) = 0. \tag{6.1}$$

This is the last typical property of the stress–energy–momentum tensor we wish to remark, which, as can be seen, is almost a tautology from Definitions 6.1 and 6.2.

In the case of relativistic fluids (Example 4) we shall have

$$(\mathrm{Eins}(L_{\mathcal{M}}))(g) = (T_g)_2(f), \qquad (\mathcal{E}_Y(L))_g(f) = 0, \tag{6.2}$$

where as we have seen

$$f^*(\mathcal{E}_Y(L))_g(f) = i_D\,\mathrm{d}(\hat{\mathcal{E}}_{TX_4}(L)(g, D)) = \mathrm{div}_g(T_g)_2(f).$$

Therefore, by Bianchi's identity $\mathrm{div}_g\,\mathrm{Eins}(L_{\mathcal{M}}) = 0$, one has that the first equation of the system (6.2) implies $f^*(\mathcal{E}_Y(L))_g(f) = 0$ and hence the second one, if $f : X_4 \to \mathcal{M}_3$ is a submersion. As is well-known, this is a classical property of relativistic fluids coupld with the gravity, according to which: *the Einstein equation for the gravitational field implies the law of motion of the matter it is coupled with.* Our result then, showing the essentiality of this fact, allows its immediate generalization to all those minimal gravitational interactions for which the second group of equations in (6.1) is equivalent to the vanishing of the divergence of the stress–energy–momentum tensor.

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